Representation Policy Iteration:
A Unified Framework for Learning Behavior and Representation

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How are we able to reason about the world across such a large dynamic range of time-scales?

How can AI systems automatically construct temporal abstractions given the uncertain nature of actions?
Credit Assignment Problem
(Minsky, Steps Toward AI, 1960)

$V(s) = V(s) + \beta[r + V(s') - V(s)]$

Challenge: a unified approach to the credit assignment problem

Temporal-difference learning
(Samuel, Barto & Sutton)
Learning to Play Checkers

(Arthur Samuel, 1950s)

``Samuel completed the first checkers program on an IBM 701, and when it was about to be demonstrated, Thomas J. Watson Sr., the founder and President of IBM, remarked that the demonstration would raise the price of IBM stock 15 points. It did." -- John McCarthy

- Samuel pioneered value function approximation using a fixed polynomial architecture.
Representation Learning by Global State Space Analysis

(Saul Amarel, 1960s)

Missionaries and Cannibal

Find symmetries and bottlenecks
Transfer of Learning:
From Task-Specific to Task-Independent Value Functions

Task-Independent Learning
- Proto-value functions
  (Mahadevan, 2005)
- Diffusion wavelets
  (Coifman and Maggioni, 2005)
- Bottlenecks, symmetries

Task-Specific Learning
- Local value functions
  (Dietterich, 2000)
- Global value functions
  (TD-Gammon, Tesauro, 1992)
- Pseudo-rewards, shaping
- Global Rewards
Representation Policy Iteration

Policy improvement  →  "Greedy" Policy

Policy evaluation

"Actor"

"Critic"

Observation of other agents

Trajectories

Proto-value functions, policy and state abstractions
LEX: Learning Symbolic Integration
(Mitchell, Utgoff, and Banerji, 1983)

\[ A_1: \int \text{poly}(x) \text{trig}(x) dx \]

"version space"

Problem Generator

\[ \int 3x^2 \sin(x) dx \]

“Actor”

Generalizer

Policy improvement

Critic

Policy evaluation

+: A_1

-: A_2

A_1

A_2
Probabilistic Finite State Models

Markov Chain

How to extend these models to capture temporal/spatial abstractions?

Hidden Markov Model

Markov Decision Process

Partially Observable Markov Decision Process
Dynamic Abstractions

- Bayes Nets
- Dynamic Bayes Nets
- Relational Markov models
- Semi-Markov Processes
- Temporal
- Relational
- Probability Distribution
- First-order Probabilistic Logic
- Factorial
- Hierarchical DBNs
- Hierarchical DPRMs
- Relational SMPs
Robot Navigation using Global Value Functions

\[ V^*(x) = \max_{a \in A(x)} \left( r(x, a) + \gamma \sum_y P_{xy}^a V^*(y) \right) \]

Choose greedy action given \( V^* \)

-1

4 10 23 49 100

Goal!

EXIT WEST
EXIT EAST
WAIT

+50
Policy Iteration
(Howard, PhD, MIT, 1959)

Policy Evaluation:
(“Critic”)
\[
V^\pi(x) = r(x, \pi(x)) + \gamma \sum_y P^\pi_{xy}(x) V^\pi(y)
\]

Policy Improvement:
(“Actor”)
\[
\pi'(x) = \arg\max_a \left( r(x, a) + \gamma \sum_y P_{xy}^a V^\pi(y) \right)
\]
Value Function Approximation

$\mathbb{R}^{|S| \times |A|}$

$\mathbb{R}^k$

LSPI [Lagoudakis and Parr, JMLR 2003]
Inverted Pendulum with Radial Basis Functions (10)
Parametric Value Function Approximation Can Fail

Approximator blind to symmetries!

Goal is to get to center of square grid

LSPI converges to an incorrect policy

Approximator blind to bottlenecks!

[Drummond, JAIR 2002]
Structural Credit Assignment: Automating Value Function Approximation

- Many approaches to value function approximation
  - Neural nets, radial basis functions, support vector machines, kernel density estimation, nearest neighbor
- How to automate the design of a function approximator?
- We want to go beyond model selection!
Proto-Value Functions

Eigenfunctions of the Laplace-Beltrami operator [Belkin and Niyogi, MLJ 2004]
Task-Level Credit Assignment: Local Value Functions
(Mahadevan and Connell, AAAI 1991; AIJ 1992)

- One approach to across-task transfer is to decompose the *global* value function into *local* value functions.
- Q-learning [Watkins, 1989] combined with a handcoded behavioral task decomposition learns much more quickly [Arkin, Brooks]

\[
Q_{M}(s, a) = (1 - \beta)Q_{M}(s, a) + \beta(r_{M}(s, a) + \max_{a'}Q_{M}(s', a'))
\]
Modeling Temporally Extended Actions
(Barto and Mahadevan, Discrete-Event Systems, 2003)

- **Semi-Markov decision process**
  - S: set of states
  - A: set of activities (or behaviors)
  - P: S x A x S x N → [0,1]: multi-step transition probability
  - R: S x A → ℝ: expected reward over duration of activity

[Visit CMU]
[Exit Hotel]
[Go to CMU]
[Exit Room]

“Exit room”

“Get Coffee”

[Kaelbling, ICML 1993]
[Parr and Russell, NIPS 1998]
[Sutton, Precup, and Singh, AIJ 1999]
[Dietterich, JAIR 2000]
How to Discover Temporal Abstractions?

- Find **bottlenecks and symmetries** in state spaces
- Rank state variables by **rate of change**
  - [Hengst, ICML 2002]
- Use **graph clustering and graph partitioning**
  - [Menache et al, ECML 2002; Simsek, Wolfe, and Barto, ICML 2005]
- Can such approaches be formalized and generalized to arbitrary (continuous or discrete) state spaces?
  - Build a model of the underlying state space manifold
  - Fourier eigenfunctions of the Laplacian [Belkin and Niyogi, MLJ 2004]
  - Diffusion wavelets based on dilations of the random walk operator on a manifold [Coifman and Maggioni, ACHA, 2005]
Proto-Value Functions

Proto-value functions are the representational building blocks of all value functions on a given environment.

Proto-value functions are based on modeling the state space manifold.
Spectral Graph Theory

- Spectral graph theory is the study of graphs using ideas from continuous manifolds [Chung, AMS 1996]
- Given an undirected graph $G = (V, E, W)$, the normalized graph Laplacian is defined as
  \[ L = D^{-1/2} (D - W) D^{-1/2} \]
  where $W$ is a weight matrix of $G$
  $D$ is the diagonal matrix of row sums of $W$
- Proto-value functions are computed by solving the equation
  \[ L f = \lambda f \]
Markov Diffusion Process

- Given an undirected graph \( G = (V, E, W) \), a Markov diffusion process is a random walk on the graph
  - Random walk is defined as \( T = D^{-1}W \)
  - \( D \) is a diagonal matrix of row sums of \( W \)

- Graph Laplacian
  - Combinatorial Laplacian: \( L = D - W \)
  - Normalized Laplacian: \( L = D^{-1/2} (D - W) D^{-1/2} = I - D^{-1/2} W D^{-1/2} \)

- Note that
  \[
  T = D^{-1}W = D^{-1/2} (D^{-1/2} W D^{-1/2}) D^{1/2} = D^{-1/2} (I - L ) D^{1/2}
  \]
How are Proto-Value Functions Learned?

Proto-value functions

Graph Laplacian
Normalized Graph Laplacian

Mountain Car Problem (2D continuous state space)

800 x 800
1500 x 1500
3000 x 3000
Representation Policy Iteration
(Mahadevan, UAI 2005)

- Learn a set of proto-value functions from a sample of transitions generated from a random walk (or from watching an expert)

- These basis functions can then be used in an approximate policy iteration algorithm, such as Least Squares Policy Iteration [Lagoudakis and Parr, JMLR 2003]
Least-Squares Policy Iteration
(Boyan, ICML 1999; Lagoudakis and Parr, JMLR 2003)

Do a random walk generating a set of transitions $D = (s_t, a_t, r, s'_t)$

\[
\tilde{A}^{t+1} = \tilde{A}^t + \phi(s_t, a_t) \left( \phi(s_t, a_t) - \gamma \phi(s'_t, \pi(s'_t)) \right)^T \\
\tilde{b}^{t+1} = \tilde{b}^t + \phi(s_t, a_t)r_t
\]

Solve the equation:

\[
\tilde{A}w^\pi = \tilde{b}
\]

\[
Q^\pi(x, a) \approx \sum_{i=1}^{k} \phi(s, a)w_i^\pi
\]
Results of RPI on a Grid World

Goal is to get to center of square grid
Results of RPI on Two Room World

Nonlinearity due to bottleneck is nicely captured by RPI!

LSPI with polynomial approximation
RPI : Inverted Pendulum Task

Normalized Graph Laplacian matrix

Sample transitions from random walk (~ 800-1600)

Proto value functions
RPI: Inverted Pendulum

Learned Proto-Value Functions  Handcoded Radial Basis Functions
Handcoded vs. Learned Representations: Inverted Pendulum

Each episode consists of a random walk till the pole is dropped.

Results show the best and worst policy learned over the 10 runs, measured in terms of the number of steps the pole remained upright.

Handcoded (RBF)  Learned (Laplacian)
Handcoded vs. Learned Representations: Inverted Pendulum

Each episode consists of a random walk till the pole is dropped.

Success is measured by measuring the percentage of times the pole is balanced upright for 3000 steps, averaged over 20 trials.

Results shown are averaged over 10 runs.

Handcoded (RBF)  Learned (Laplacian)
Learned vs. Handcoded Representations: Chain MDP

Results averaged over 5 runs

Convergence time
Policy error

Laplacian
RBF
Poly
Multilevel Proto-Value Functions using Diffusion Wavelets

(Coifman and Maggioni, ACHA 2005; Maggioni and Mahadevan, 2005)

- Laplacian eigenfunctions are global Fourier basis functions over the state space.
- Wavelets are another popular tool for harmonic analysis that use compact basis functions.
- **Diffusion wavelets** generalize standard wavelets to graphs and manifolds.
- They provide a formal basis for multilevel proto-value functions, as well as hierarchical abstraction of Markov processes on graphs.
Learning Proto-Value Functions using Diffusion Wavelets
(Mahadevan and Maggioni, 2005)

Level 1  Level 4  Level 7

Two room environment
Learning Proto-Value Functions using Diffusion Wavelets
Learning Diffusion Wavelets

- Diffusion wavelets are constructed using a layered spectral analysis of reversible random operator $T$
- The key idea is that powers of $T$ have faster spectral decay than $T$, and require a smaller set of basis functions
- The algorithm takes as input a precision parameter $\varepsilon$
- It returns a diffusion wavelet tree that constructs a hierarchical layered set of representations of $T$ at varying levels of abstraction
Multiscale Analysis of Random Walks using Diffusion Wavelets
(Maggioni and Mahadevan, U.Mass TR 2005)

Sampling of a continuous two-room environment
Abstracting Random Walks using Diffusion Wavelets

Level 3

Basis functions at each level

Level 6

Spectrum

Level 9

Inverted pendulum

Random walk

28 x 28

270 x 270

80 x 80
Abstracting Random Walks using Diffusion Wavelets

Mountain Car

Level 2
600 x 600

Level 4
55 x 55

Level 6
10 x 10

Level 8
5x5

Position

Velocity
A Faster Critic: Policy Evaluation with Diffusion Wavelets
(Maggioni and Mahadevan, U.Mass TR 2005)

\[ V^\pi = R^\pi + \gamma P^\pi V^\pi = (I - \gamma P^\pi)^{-1} R^\pi \]

- Traditionally, policy evaluation requires solving a system of ISI linear equations, which takes \(O(ISI^3)\)
- By constructing a diffusion wavelet tree from the transition matrix, it is possible to develop a significantly faster policy evaluation algorithm that runs in \(O(ISI \log^2 ISI)\)
  - Precomputation phase: \textit{task-independent}
  - Inversion step: \textit{reward-specific}
Direct Policy Evaluation with Diffusion Wavelets
Summary: Proto-Value Functions

- A major challenge facing AI research is to design a unified framework for learning representation and behavior
  - Representation Policy Iteration
- Proto-value functions
  - Fourier and wavelet representations that capture large-scale geometry of the state space manifold
  - Provides a unified solution to the problem of temporal, structural, and task-level credit assignment
- Extensions of proto-value functions
  - Factored Markov decision processes [Boutilier, Guestrin]
  - Relational domains [Getoor, Jensen, Koller]
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