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A Note on Pattern Reproduction in Tessellation Structures*

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Tessellation structures that reproduce arbitrary patterns are special cases of tessellation structures having local transformations that are linear operators. We introduce a novel formulation of tessellation structures which emphasizes the connection between these structures and concepts of functional analysis. Using this formulation a behavioral analysis technique is developed which implies the earlier results on pattern reproduction and generalizes them to tessellation structures whose state alphabets are arbitrary fields of non-zero characteristic and whose tessellation arrays are arbitrary countable abelian groups. It is also shown that a local transformation can be chosen to produce at a specified time any desired set of "copies" of an initial pattern each multiplied by a specified scalar. We then indicate that connections exist between linear tessellation structures and linear partial differential equations which describe wave propagation by giving an example of a classical form of pattern reproduction.

1. INTRODUCTION

Amoroso and Cooper [2] have shown that for an arbitrary state alphabet A, one and two-dimensional tessellation structures can be constructed that have the ability to reproduce any finite pattern contained in the tessellation space, and that if A has a prime number of elements, the copies will appear in quiescent environments. Ostrand [14] extended this result to tessellation structures of any finite dimension. Hamilton and Mertens [9] showed that similar pattern reproduction is possible for tessellation structures with arbitrary finite neighborhood indices. Anderson [3] showed how these results, when restricted to the case of reproduction in quiescent environments, can be proved form an elementary fact about polynomial multiplication for polynomials with coefficients in the group of integers $\{0, 1,..., p-1\}$ with modulo p addition for p a prime.

Although the original study by Amoroso and Cooper [2] may have been motivated by possible biological applications, the pattern reproducing tessellation structure they presented was not constructed to model any specific biological process, nor was it argued that the tessellation structure's mechanism was biologically suggestive. If this mechanism is therefore viewed as a purely mathematical construction, then it is pertinent to ask

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whether it is merely an interesting mathematical curiosity or whether it is connected in some way with more general and better known mathematical structures. In this note we show how pattern reproducing tessellation structures can be viewed as special members of the class of tessellation structures having local transformations that are linear operators. We then develop a behavioral analysis method for members of this class that have local transformations that are linear over fields of non-zero characteristic. (The characteristic of a field is the least positive integer p such that $pa = 0$ for every element a of the field. If no such positive integer exists, the field is said to have characteristic 0.) We then show that the results in Refs [2, 3, 9, 14], when restricted to pattern reproduction in quiescent environments, follow very naturally using this method.

We also show that it is possible to specify a tessellation structure, for any state alphabet that is a finite field, which can create all the scalar multiples of a given pattern. Thus, a pattern's "offspring" need not be identical copies of the original. These results are shown to hold for tessellation structures having arbitrary countable abelian groups as tessellation arrays. We feel that this generality is not worthwhile for its own sake but rather for its explanatory value.

We begin by developing, for use throughout this article, a formulation of tessellation structures that is equivalent to but slightly different from that used by Amoroso *et al.* in that it is influenced by notational conventions used in functional analysis. It is not our intention to merely augment the already substantial number of different notations and terminologies associated with the concept of tessellation structures. However, besides making our results easy to prove, the merit of our formulation is that it simplifies notation and, more importantly, permits straightforward generalization in directions which make contact with standard mathematical topics. This formulation, we feel, can contribute substantially to the further study of tessellation structures.

2. Notation and Definitions

Let Z^d denote the set of all d-tuples of integers, and let A be a set with distinguished element denoted by 0. The *support* of a function $c: Z^d \to A$ is the set $X \subset Z^d$ such that $c(\xi) \neq 0$ if and only if $\xi \in X$. A function $c: Z^d \to A$ has *finite support* if its support is a finite set. Let C_F denote the set of all functions from Z^d to A that have finite support. For any subset X of Z^d and any $c \in C_F$, let $c \mid X$ denote the *restriction of c to X*, i.e., $c \mid X: X \to A$ is given by $(c \mid X)(\xi) = c(\xi)$ for $\xi \in X$. For $x, y \in Z^d$, $x + y$ will denote the componentwise sum of x and y. Z^d with this operation is an abelian group.

The major difference between the formulation to be given here and that given in [2] is our use of the notion of a shift operator. For every $\xi \in Z^d$ define a *shift operator* L_ξ mapping C_F to C_F such that

$$(L_\xi c)(x) = c(\xi + x)$$

for every $c \in C_F$ and $x \in Z^d$. Note that since $L_x L_y = L_{x+y}$, the set of shift operators forms a group, with operator composition as the binary operation, which is isomorphic to the group Z^d.

A *tessellation structure* is defined in [2] as a 4-tuple $M = (A, Z^d, X, \tau)$ where A is an arbitrary finite set called the *state alphabet*, Z^d is the *tessellation array*, X is a finite subset of Z^d called the *neighborhood index*, and τ is the *parallel transformation*. Intuitively, at any discrete time step, to each point in Z^d there corresponds a symbol from the state alphabet A. This infinite array of symbols is an array configuration and is formalized as a function $c: Z^d \to A$. Each point in Z^d is taken to label a cell. At the next discrete time step a new array configuration appears which is the result of the uniform application of a *local transformation* at each cell which determines the next symbol for that cell from the present symbols of its neighboring cells. The uniform and simultaneous application of the local transformation at each cell results in a parallel transformation τ, a mapping whose range and domain is the set of possible array configurations. If one restricts array configurations to those having finite support, τ should map C_F to C_F.

We formally define such a tessellation structure $M = (A, Z^d, X, \tau)$ by first letting $\sigma: C_F \to A$ be any mapping with the following two properties:

(i) if $c \in C_F$ is identically equal to 0, then $\sigma(c) = 0$; and

(ii) if $c_1, c_2 \in C_F$ are such that $c_1 \mid X = c_2 \mid X$, then $\sigma(c_1) = \sigma(c_2)$.

The map σ is the local transformation. Condition (i) will insure that τ is a map from C_F to C_F, i.e., that configurations with finite support will always be followed by configurations also having finite support. Condition (ii) is another way of saying that σ is a *local* transformation: its action does not depend on a function's value outside of the finite set X. Then we define the parallel transformation $\tau: C_F \to C_F$ to be given for any $c \in C_F$ by

$$\tau(c) = c'$$

where (1)

$$c'(\xi) = \sigma L_\xi(c) \qquad \text{for all} \quad \xi \in Z^d.$$

The next state of cell ξ is the result of the local transformation σ acting on the shift-by-ξ of the current configuration c. It can be verified that this construction is equivalent to that given in [2].

3. Linear Tessellation Structures

We will call a tessellation structure a *linear tessellation structure* (LTS) if the state alphabet A is a field (with the distinguished zero element the zero element of the field) and the local map σ, in addition to having property (ii) above, is a *linear* map from C_F to the field A. This means that for any two configurations c_1 and c_2 in C_F and any "scalar" $a \in A$ that $\sigma(c_1 + c_2) = \sigma c_1 + \sigma c_2$ and $\sigma(ac_1) = a\sigma(c_1)$. Addition of configurations is taken to be pointwise function addition (i.e., $(c_1 + c_2)(\xi) = c_1(\xi) + c_2(\xi)$) so that C_F can be regarded as a vector space over the field A. A linear map from a vector space to its underlying field is called a linear functional. Thus an LTS is a tessellation structure whose state alphabet is a field and whose local transformation is a linear functional which satisfies condition (ii) above. Condition (i) automatically holds because of the zero preserving property of linear maps. Linear tessellation structures are special kinds of what are called in [4] *linear cellular automata* which are defined for an arbitrary countable group instead of Z^d and for state alphabet A a finite dimensional vector space. Similar structures are studied in [8, 11].

Since σ satisfies condition (i) and (ii) above and is a linear map, it is easy to see that it can be written as a particular weighted sum of the configuration values over the set X, i.e., there is a function $W: Z^d \to A$ with finite support X (so that $W \in C_F$) which represents the linear functional σ as follows:

$$\sigma(c) = \sum_{x \in Z^d} W(x)\, c(x) = \sum_{x \in X} W(x)\, c(x). \tag{2}$$

Summing over just the set X is possible since the support of W is X. In other words, $\sigma(c)$ is the inner product of the vectors W and c. The questions surrounding the representation of linear functionals on general vector spaces as inner products is central to much functional analysis. In the framework adopted here it is possible, given any functional on C_F satisfying condition (ii), to find a vector $W \in C_F$ which represents that functional as in Eq. (2). However, this kind of representation is not generally possible in arbitrary infinite dimensional vector spaces.

By using Eq. (2) in Eq. (1), it follows that

$$\begin{aligned}(\tau(c))(\xi) = \sigma L_\xi(c) &= \sum_{x \in X} W(x)(L_\xi c)(x) \\ &= \sum_{x \in X} W(x)\, c(\xi + x) \qquad (3)\\ \text{for all} \quad c \in C_F \quad &\text{and} \quad \xi \in Z^d.\end{aligned}$$

You may recognize that τ is a form of cross correlation between array configurations c and the fixed function $W \in C_F$ which represents the local transformation τ. This occurs, for example, in time series analysis where the field A is the real field and the tessellation array is simply Z whose elements represent discrete points in time rather than space. The cross correlation is used to study the relationship between two time series. See, for example [12].

Letting $-X = \{-x \mid x \in X\}$, note that Eq. (3) can be written

$$(\tau(c))(\xi) = \sum_{x \in -X} W(-x)\, c(\xi - x).$$

If we define $K \in C_F$ by $K(x) = W(-x)$ for all $x \in Z^d$, then this can be written as the following convolution sum:

$$(\tau(c))(\xi) = \sum_{x \in -X} K(x)\, c(\xi - x). \tag{4}$$

The function K can be thought of as the *impulse response function* of the tessellation structure since if the initial array configuration is an "impulse" at the origin of Z^d (i.e., the cell at the origin is in state 1 and the other cells are quiescent), then the next configuration will be K (i.e., K is the system's response to the impulse). A transition in an LTS is a convolution of the current configuration with the impulse response function. Except for the higher dimensionality and spatial rather than temporal interpretation of the array, this situation is identical to that for time-invariant linear systems. The uniformity of the tessellation array implies spatial-invariance of the parallel transformation. See [4, 5, 8, 15]. Since polynomial multiplication can be viewed as a special type of convolution, (see esp. [1]), our approach specializes to that given in [3].

Finally, we'll find it convenient to write τ as a linear combination of shift operators. Since Z^d is an abelian group, $(L_y c)(x) = (L_x c)(y)$ for all $x, y \in Z^d$ and τ can be written as follows:

$$\tau = \sum_{x \in X} W(x) L_x . \tag{5}$$

Each of the tessellation structures given in [2], [9], and [14] that reproduces patterns so that the copies are in quiescent environments is a special kind of LTS. The state alphabet $A = \{0, 1,..., p - 1\}$, for p a prime and with modulo p addition and multiplication, is a finite field ($GF(p)$). The one-dimensional structure constructed by Amoroso and Cooper is an LTS with local map σ represented by the function W given by:

$$\begin{aligned} W(0) &= 1, \\ W(-1) &= 1, \\ W(\xi) &= 0 \qquad \text{for} \quad \xi \notin \{0, -1\}. \end{aligned} \tag{6}$$

This LTS is shown schematically in Fig. 1. Only the non-zero coefficients (here each

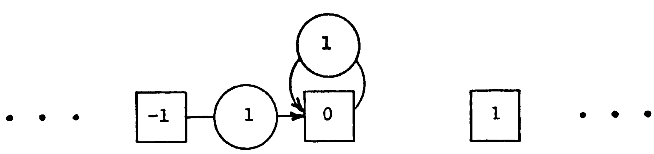

FIG. 1. A schematic representation of a one-dimensional LTS that reproduces arbitrary patterns.

equal to 1) are shown, and it is understood that this interconnection pattern is repeated uniformly for every cell. In this case the values of W are either 0 or 1, but the result proved in the next section holds for W with finite support and arbitrary values in A.

More familiar examples of LTSs occur when the field $A = R$, the real numbers, although technically they are not tessellation structures since A would not be finite. Such structures are more commonly known as multi-dimensional linear difference equations and are often used to approximate solutions to partial differential equations. Other examples, again with $A = R$, model random walks where configurations are interpreted as probability distributions (see [13]). In this case W must also be a probability distribution.

4. Neighborhood Decomposition

In this section we develop a method for analyzing the behavior of any tessellation structure which is linear over any field of non-zero characteristic and then show how this is related to pattern reproduction. Suppose $M = (A, Z^d, X, \tau)$ is an LTS where the local transformation σ is represented by a function $W: Z^d \to A$ with finite support X. Define for each $x \in X$ the function $W_x: Z^d \to A$ by

$$W_x(\xi) = \begin{cases} W(x) & \text{for } \xi = x, \\ 0 & \text{otherwise.} \end{cases}$$

Each W_x has just one non-zero value. For each $x \in X$, let M_x be the LTS $(A, Z^d, \{x\}, \tau_x)$ where τ_x is determined by a local map σ_x represented by W_x, i.e., using Eq. (2)

$$\sigma_x(c) = \sum_{\xi \in \{x\}} W_x(\xi)\, c(\xi) = W(x)\, c(x).$$

Each M_x is an LTS in which each cell has a single neighbor. Thus, any LTS M with neighborhood index X is associated with a family $\{M_x \mid x \in X\}$ of LTSs each having a one-element neighborhood index. Figure 2 shows M_0 and M_{-1} associated with the

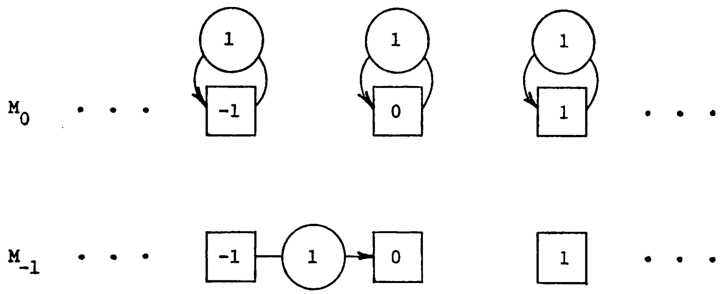

FIG. 2. The LTSs M_0 and M_{-1} associated with the LTS shown in Fig. 1.

LTS of Fig. 1. Note that for each M_x, $x \in X$, the parallel transformation τ_x can be written in operator form according to Eq. (5) as

$$\tau_x = W(x) L_x . \tag{7}$$

The linearity of the parallel transformation τ of M implies that for a single transition it is possible to compute $\tau(c)$ by computing $\tau_x(c)$ for each $x \in X$ and superimposing (pointwise adding) the resulting configurations, i.e.,

$$\tau(c) = \sum_{x \in X} \tau_x(c) \qquad \text{for any} \quad c \in C_F$$

obtained by substituting Eq. (7) in Eq. (5). However, this is not true for more than a single transition. It is not generally possible, even with linearity, to run the M_x separately for n time steps, each having initial configuration c, and then superimpose the final configurations to arrive at what M would have computed in n time steps from initial configuration c. Under certain conditions, however, it is possible:

THEOREM. *Let M be an* LTS (A, Z^d, X, τ) *where A has characteristic $p > 0$. With the* LTSs M_x, $x \in X$, *defined as above, it is true that*

$$\tau^{p^m} = \sum_{x \in X} \tau_x^{p^m} \qquad \text{for each non-negative integer } m.$$

Proof. The binomial theorem implies for $a, b \in A$, where A is a field of characteristic $p > 0$, that for $m > 0$

$$(a + b)^{p^m} = a^{p^m} + b^{p^m} \tag{8}$$

(see [16]). This is clearly also true for any finite sum. From Eq. (5) we have that

$$\tau^{p^m} = \Big(\sum_{x \in X} W(x) L_x\Big)^{p^m} \tag{9}$$

If the shift operators were not present then Eq. (8) would be immediately applicable. As it stands, though, (9) is not a multiplication of sums in the field A, but composition of linear combinations of shift operators. But note that since Z^d is an abelian group

$$(W(x) L_x)(W(y) L_y) = W(x)\, W(y) L_x L_y$$

and

$$L_x L_y = L_y L_x \qquad \text{for all} \quad x, y \in Z^d.$$

Thus the coefficients $W(x)$ will group together in the expansion of (9) in the same way they would if the shift operators were not there. Therefore (8) implies that

$$\tau^{p^m} = \sum_{x \in X} [(W(x) L_x)^{p^m}]$$

or

$$\tau^{p^m} = \sum_{x \in X} \tau_x^{p^m}. \qquad \text{Q.E.D.}$$

The subscripts of the shift operators behave like the exponents in the polynomial formulation in [3], that is, they follow the additive structure of Z^d. From a more abstract point of view, since τ is a linear combination of shift operators which form a group isomorphic to Z^d, τ is an element of the group algebra of Z^d over A [6]. Moreover, one should note that the only property of the tessellation array Z^d which is used for the theorem is that it is an abelian group. Thus this result can be extended to tessellation arrays which are abelian groups other than Z^d, including finite groups corresponding to toroidal arrays of cells.

Whenever this theorem applies, it is easy to deduce the form of the configurations $\tau^{p^m}(c)$ for any initial configuration $c \in C_F$ and $m = 0, 1, 2,\ldots$. One constructs the tessellation structures M_x , $x \in X$, determines $\tau_x^{p^m}(c)$, $x \in X$, and then superimposes these final configurations. The power of this result lies in the fact that since each cell in each M_x has only one neighbor, it is very easy to find $\tau_x^{p^m}(c)$ for any c. In the structure representing M_x there will be no feedback loops so that configurations will unidirectionally propagate with "amplitude" changing depending on the constant $W(x)$, i.e.,

$$\tau_x^{p^m}(c) = (W(x) L_x)^{p^m} (c) = (W(X))^{p^m} L_{p^m \cdot x}(c) \tag{10}$$

where $p^m \cdot x$ means $x + x + \cdots + x$ (p^m times) and "+" is addition in Z^d. If $W(x) = 1$, then $(W(x))^{p^m} = 1$ so that by Eq. 10 $\tau_x^{p^m}$ is the simple translation $L_{p^m \cdot x}$ obtained by performing the translation L_x p^m times in succession. The theorem says that for time steps p^m, $m = 0, 1, 2,\ldots$, the configuration of M will be the superposition of these translated versions of the initial configuration (cf. the concept of dilated neighborhood in [3]).

5. Pattern Reproduction

The Amoroso–Cooper result for one-dimensional pattern reproduction in quiescent environments can be seen to follow from the neighborhood decomposition method. Since the field A consisting of the integers $0, 1,\ldots, p - 1$ with modulo p addition and multiplication has characteristic $p > 0$, the LTSs M_0 and M_{-1} shown in Fig. 2 can be run separately each starting with a configuration c. For any $m > 0$, their configurations after p^m time steps can be superimposed to produce what M of Fig. 1 produces in p^m time steps. M_0 does nothing to c since τ_0 is the identity operator. M_{-1} computes $(1^{p^m}) L_{-p^m} = L_{-p^m}$, i.e., it shifts c to the right p^m cells. Thus, m is chosen large enough to insure that the translation L_{-p^m} shifts a pattern far enough to the right so that there is no overlap of its support after the shift and its original support (i.e., $\sup c \cap \sup L_{-p^m} c = \varnothing$ where c is the configuration containing a pattern and $\sup c$ is the support of c). For such an m, the superposition of the configurations after p^m time steps of M_0 and M_{-1} is a configuration containing two copies, each in a quiescent environment, of the pattern in c. One copy is located at its original position (due to M_0), the other is located p^m cells to the right (due to M_{-1}).

The d-dimensional case in [14] can be analyzed analogously. The result of Hamilton and Mertens [9] (restricted to the case of quiescent environments) can also be deduced from the theorem. However, the arbitrary neighborhood index makes it more difficult to specify how large m needs to be for there to be no overlap between the scattering copies of the initial pattern. It's not hard to see, though, that for m large enough, copies shifted by a set of shift operators $L_{p^m \cdot x}$, $x \in X$, will "scatter" sufficiently for mutual overlap to be removed.

Note that all of these LTSs have local transformations specified by functions W whose non-zero values are all 1. For other values, Eq. (10) implies that at times p^m each of the scattering "copies" of the initial pattern will be the original pattern uniformly multiplied by some scalar which may be different for each copy. Thus it is not only possible to construct a tessellation structure which can reproduce arbitrary patterns, but also one that can produce "copies" each of which is a preselected scalar multiple of the original pattern.

To be specific we'll describe one way of synthesizing such a tessellation structure. Suppose $M = (A, Z^d, X, \tau)$ is a LTS whose local transformation σ is represented by a function $W: Z^d \to A$ with finite support X. Let A be a finite field of characteristic p and let $\{M_x \mid x \in X\}$ be the associated family of LTSs having one-element neighborhood indices. It is a fact about finite fields (see, for example, [16]) that every finite field A of characteristic p has p^n elements for some non-negative integer n, and that $a^{p^n} = a$ for all $a \in A$. Then $a^{p^{mn}} = a^{p^n \cdot p^n \cdots p^n}$ (p^n appearing m times) $= a$. Thus Eq. 10 implies for each $x \in X$ that

$$\tau_x^{p^{mn}} = (W(x))^{p^{mn}} L_{p^{mn} \cdot x} = W(x) L_{p^{mn} \cdot x}$$

$m = 0, 1, 2, \ldots$. This means that after p^{mn} time steps the configuration of each LTS M_x is the shift $L_{p^{mn} \cdot x}$ of the initial configuration multiplied by the scalar $W(x)$. The theorem says that for $m = 0, 1, 2, \ldots$ these resultant configurations can be superimposed to produce what the LTS M would have produced. Choosing the values $W(x)$, $x \in X$, to be the desired scalars, the LTS M therefore produces "copies" of any initial pattern each multiplied by a scalar used to specify M's local transformation. One must wait p^{mn} time steps where m is large enough to eliminate pattern overlap.

For example, the LTS (A, Z, X, τ) shown in Fig. 3 with $A = GF(3)$ will produce

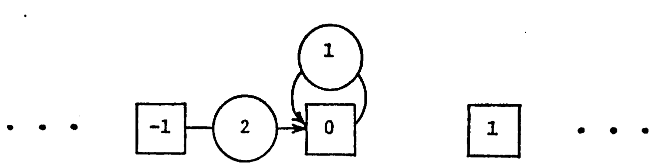

FIG. 3. An LTS over $GF(3)$ which produces scalar multiples of arbitrary patterns.

a configuration containing any original pattern and a pattern whose cell states are 2 times (modulo 3 multiplication) the corresponding states in the original pattern. In this case, the second pattern is a modulo 3 "negative" of the original since $2a + a = 0$ (modulo 3). One must wait $p^{m \cdot 1}$ time steps for m large enough to eliminate overlap.

Finally, we note that these results extend to tessellation arrays which are abelian groups other than Z^d since only the abelian group properties of Z^d have thus far been used. However, in the case of tessellation arrays specified by finite groups an obvious caveat is needed: a pattern may be too large for it to be possible to remove overlap between its copies.

6. Concluding Remarks

Yamada and Amoroso [18] remark that if Z^d were replaced by R^d, where R denotes the reals, then a tessellation structure might be viewed as a model of a continuous physical system with the local transformation represented in terms of integration rather than summation. The notation used in this note is readily extendible to the continuous case since if the tessellation array Z^d is replaced by the Euclidean space R^d, where R denotes the set of real numbers, the notions of function restriction and shift operator work equally well. The definition of σ and τ (Eq. (1)) remain unchanged with the exception that the neighborhood index X might be permitted to be any closed and bounded subset of R^d. This is more fully described by the author in [4].

We mention the extension to the continuous case since the class of linear tessellation structures as defined in this note are mathematically analogous to continuous structures specified by certain kinds of linear partial differential equations which describe wave propagation. This is not surprising since it seems certain that von Neumann's original formulation of the tessellation structure (or cellular automaton) concept was motivated by a knowledge of partial differential equations (see [17]). What seems interesting with regard to pattern reproduction is that a system described by the classical one-dimensional wave equation also reproduces arbitrary initial patterns. If $c(x, t)$ denotes, for example, the displacement at time t and position x along an infinitely long vibrating string, then the wave equation takes the form

$$\frac{\partial^2 c}{\partial t^2} = k^2 \frac{\partial^2 c}{\partial x^2}$$

where k depends on the tension and density of the string. For initial configuration $c(x, 0) = f(x)$ (having no initial velocity) the solution is (see, for example [7]):

$$c(x, t) = 1/2[f(x + kt) + f(x - kt)]$$

The term $f(x + kt)$ represents the initial configuration shifted kt units to the left; the other term represents this configuration shifted kt units to the right. Thus, any initial configuration splits into two parts, each maintaining the shape of the original, which propagate in opposite directions. If the initial configuration has bounded support, after an appropriately long time there will be two "copies" of the initial pattern each having one-half the height of the original.

Wave propagation thus involves a form of pattern reproduction although it is not customary to think in these terms. Similarly, it is appropriate to view the behavior of the linear tessellation structures described above as discrete forms of wave propagation which exhibit complex interference properties due to the non-zero characteristic of the underlying field.

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