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Minimum Principles in Motor Control

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Minimum (or minimal) principles are mathematical laws that were first used in physics: Hamilton's principle and Fermat's principle of least time are two famous example. In the past decade, a number of motor control theories have been proposed that are formally of the same kind as the minimum principles of physics, and some of these have been quite successful at predicting motor performance in a variety of tasks. The present paper provides a comprehensive review of this work. Particular attention is given to the relation between minimum theories in motor control and those used in other disciplines. Other issues around which the review is organized include: (1) the relation between minimum principles and structural models of motor planning and motor control, (2) the empirically-driven development of minimum principles and the danger of circular theorizing, and (3) the design of critical tests for minimum theories. Some perspectives for future research are discussed in the concluding section of the paper.

Motor control tasks tend to be ill-defined in the sense that task requirements can generally be met by a large (or infinite) number of different movements. Bernstein (1967) was the first to draw attention to this issue, and Saltzman (1979) has treated it in considerable detail. For illustration, consider a simple point-to-point reaching task. In such a task, the trajectory of the arm (here defined as the time evolution of arm configurations) is ill-defined on three levels of description. First, the tip of the arm (i.e., the hand) may approach the target along any arbitrary geometric curve (or hand path) as long as that curve connects the initial and target hand locations; second, the progress of the hand along any such path may be timed in an arbitrary fashion as long as overall constraints on movement duration are satisfied; and third, each spatial location of the hand may generally be achieved by an infinite number of joint-angle configurations (arm postures).

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These indeterminacies would not be of particular interest to the motor control researcher if they were resolved randomly, i.e., if people arbitrarily selected movements from the set of those that are compatible with the task requirements. But this is not the case: Experimental studies have consistently found that the kinematics of arm movements are highly stereotypical under a large variety of experimental conditions. For instance, the velocity profiles of point-to-point movements of various amplitudes and durations are consistently bell-shaped, single-peaked, and approximately symmetrical (Abend, Bizzi, & Morasso, 1982; Atkeson & Hollerbach, 1985; Morasso, 1981, 1983); they generally vary little within and between subjects (e.g., Miall & Haggard, 1995), and variability tends to decrease with practice (Georgopoulos, Kalaska, & Massey, 1981).

One explanation of this behavioral uniformity is that certain movements are systematically preferred over others because they satisfy some efficiency criterion (such as the minimization of movement time or of metabolic energy requirements) that is common to all people. In this view, alternative movements are ranked according to the cost that their execution is expected to incur and movement selection is understood as a process of cost minimization. Theories of this kind (we shall refer to them as minimum theories) have become quite influential in the field of motor control due to the landmark works of Hogan (1984a), Flash and Hogan (1985), and Uno, Kawato, and Suzuki (1989), which have been followed by many others.

The objective of the present paper is to provide an integrative review of the contributions that minimum theories have made to our understanding of motor behavior. The paper is organized as follows. First, the general theoretical approach and some of the basic mathematical tools are introduced. This is done by discussing the classic use of minimum principles in mechanics, more recent applications in biology and engineering, and the relation of these to the use of minimum principles in motor control. Second, a comprehensive overview of the work in motor control is provided. Minimum principles in motor control are usually motivated by indeterminacies that arise at various stages of a sensorimotor transformation. They may therefore be classified based on the particular indeterminacy they are intended to resolve, and the overview is organized along the lines of this taxonomy. Third, we address several issues of broader concern such as the danger of circular theorizing associated with the empirically-driven development of minimum principles and the problem of designing critical tests for minimum theories. Fourth, in the concluding section of the paper, some perspectives for future research are discussed.

APPLICATIONS IN MOTOR CONTROL AND RELATED SCIENCES

Minimum theories are not unique to the field of motor control but play important roles in a variety of sciences including physics (particularly mechanics), chemistry, biology, economics, and engineering (for a review, see Schoemaker, 1991). The application of minimum principles to motor control is most closely related to their use in mechanics, evolutionary biology, and engineering. Below, we shall briefly review the use of minimum principles in these three disciplines and compare it to their role in motor control.

Mechanics

Mechanics is the study of the laws that govern the movement of dynamical systems. One type of law that can be used to describe the motion of such a system is the functional relation, an expression of the form $\mathbf{x} = \mathbf{f}(t)$, where $\mathbf{x}$ denotes the configuration of the system (i.e., the position and orientation of all the system components), and t denotes time. We may, for instance, study the free fall of a point mass and find that its motion is well described by the expression

$$x(t) = x_0 + \dot{x}_0 t - \tfrac{1}{2} g t^2, \tag{1}$$

where g is the gravitational acceleration near the surface of the earth, x_0 denotes the mass's initial position (height), and $\dot{x}_0$ denotes its velocity. Similarly, we may study the motion of a mass attached to a (frictionless and massless) linear spring and find that it obeys

$$x(t) = x_{eq} + A \cos \omega t + B \sin \omega t, \qquad \omega \equiv \sqrt{k/m}, \tag{2}$$

where x_{eq} is the equilibrium point of the spring, k is the spring constant, A and B are constant coefficients whose values depend on the initial position and velocity of the spring, and m is the mass.

The problem with this type of approach is that we obtain a different law for each type of motion. In mechanics, we know that such a diversity of laws is unnecessary: At least theoretically, all phenomena of classical mechanics may be derived from Newton's laws. (Note, though, that Newton's second law is not a functional relation but a differential one.) Alternatively, they may also be derived from a single unifying law known as

Hamilton's principle[1] . This law is not a functional relation, nor is it a differential one; it is a minimum (or, more precisely, a variational) principle. It states that the trajectory of any dynamical system is such that it extremizes (i.e., minimizes or maximizes) the time integral of the scalar quantity $L \equiv T - U$, where T and U are the kinetic and potential energies of the dynamical system, respectively.

For the point mass discussed above, the kinetic energy is $T = \frac{1}{2} m \dot{x}^2$, and its potential energy is $U = mgx$. According to Hamilton's principle, the trajectory of the point mass is some function $x(t)$ that has the property that it extremizes the integral

$$C[x] = \int_{t_0}^{t_f} L(x(t), \dot{x}(t))\, dt. \tag{3}$$

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[1] Hamilton's principle was formulated by Sir W. R. Hamilton (1805–1865), Irish mathematician and astronomer. It is central to analytical mechanics, and variations of it have also been applied to a number of other domains of physics. For instance, one of the fundamental features of the equations that underly the modern theories of general relativity and quantum mechanics is their self-adjoint character, which means that they are derivable from a variational principle. Two excellent sources for further study of the variational principles of physics are Gossick (1967) and Lanczos (1970).

The appropriate methods for finding $x(t)$ are provided by the *calculus of variations*[2], an extension of the calculus to optimization problems that have a function rather than a variable as the unknown quantity. Fundamental to the calculus of variations is the Euler–Poisson equation

$$\sum_{k=0}^{n} (-1)^k \frac{d^k}{dt^k} \frac{\partial f(x, \dot{x}, ..., x^{(k)})}{\partial x^{(k)}} = 0, \tag{4}$$

where $x^{(k)}$ denotes the kth time derivative of x and, in our particular case, $n = 1$ and $f = L(x, \dot{x})$. This equation provides a necessary condition $x(t)$ must satisfy if $C[x]$ is to attain a minimum or maximum. Using our definitions of the kinetic and potential energy and inserting these into Eq. (4), we have $\ddot{x} = g$, and after integrating this twice, we obtain Eq. (1), the equation of motion of the point mass. For the spring, we may proceed in the same fashion. The kinetic energy of the attached mass is $T = \frac{1}{2}m\dot{x}^2$, and its potential energy is $U = \frac{1}{2}k(x - x_{eq})^2$. Application of the Euler–Poisson equation now yields the ordinary second-order linear differential equation

$$\ddot{y} + \omega^2 y = 0, \qquad y \equiv x - x_{eq}, \tag{5}$$

the solution of which is given by Eq. (2).

Hamilton's principle is extremely powerful. It not only allows us to describe the motion of a point mass or a linear spring, but (as Newton's laws) applies to any mechanical system, no matter what its complexity. It is not an alternative to the Newtonian theory of mechanics, which we know is quite correct; indeed, it is entirely consistent with that theory[3]. It does, however, provide an alternative device for dealing with complicated mechanical problems that is often superior to the Newtionian approach. It has the advantage of not requiring any knowledge of the forces among the particles of which a mechanical system is composed. It also has the advantage that its application leads to equations of motion that are invariant under any coordinate transformation (which results from the statement of the principle being independent of any special coordinate system). This permits the formulation of mechanical problems in terms of generalized coordinates, which greatly facilitates the representation of constraints among the particles of a dynamical system.

Hamilton's principle does not, however, offer any mechanistic explanation for the behavior of dynamical systems, except for the absurd one that each such system is endowed with a form of intelligence that computes optimal trajectories. However, contrary to popular belief, many other physical laws do not have a mechanistic

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[2] An introduction to the calculus of variations can be found in Dreyfus (1965), and in-dept treatments are provided by Akhiezer (1962) and Bliss (1946). For a history of the calculus of variations, see Goldstine (1980).

[3] For complex systems, this is not always obvious because the structure of the obtained results is typically quite different for the two approaches. The results are nonetheless equivalent, as can usually be shown by transforming the result obtained with one approach into the result obtained with the other. Silver (1982) discusses this in detail for the dynamics of open-loop kinematic chains.

explanation either. Consider, for instance, the law of universal gravitation. When Newton first proposed it, many of his contemporaries rejected it because they could not conceive of a mechanism that would allow one object to act on another across a vacuum (cf., Kline, 1962, Chap. 17), Today, we still do not know of any such mechanism, but physicists have long accepted that all we may obtain from this and other laws are quantitative mathematical descriptions of physical phenomena rather than mechanistic explanations (cf., Feynman, Leighton, & Sands, 1963, Chap. 7; Kline, 1962, Chap. 17).

Hamilton's Principle and other minimum principles in physics should therefore be regarded *descriptive* laws and not as explanatory ones. In evolutionary biology, the situation is a different one. We shall consider this field next.

Evolutionary Biology

Since the early 1970s, evolutionary biologists have increasingly invoked minimum (or optimality) principles in their study of such diverse types of animal behavior as patterns of locomotion (Alexander, 1980, 1989), foraging (Kamil, Krebs, & Pulliam, 1987; Krebs, 1984), competition (Maynard Smith, 1974, 1982), cooperation (Axelrod, 1984; Axelrod & Hamilton, 1981) and reproductive behavior (Daly & Wilson, 1983), The general motivation behind this research is the idea that the process of natural selection continuously improves the genetic design of animals with respect to traits that are closely related to the animals' reproductive success. It is believed that in many instances these improvements have actually led to optimal designs.

A good example of the general approach is Krebs, Kacelnik, and Taylor's (1978) study of the foraging behavior of the great tit, a small insect-eating bird. In that study, a number of these birds were given the opportunity to operate two machines that dispensed units of food with different but unknown probabilities. The question was whether the birds were able to sample the two machines in a way that would maximize their total food intake (or equivalently would minimize their feeding time per unit of food). To do so, the birds had to solve an instance of what statisticians call a bandit problem[4] . Ideal solutions to this type of problem involve an optimal combination of random exploration (to estimate the different reward probabilities) and exploitation of what appears to be the more promising action. Interestingly, the birds' feeding strategy was found to closely approximate the optimal one: an impressive finding, considering that the mathematical determination of such an optimal strategy is extremely challenging and requires extensive numerical computations.

Unlike physicists, biologists are generally not satisfied to view minimum principles as purely mathematical laws whose main purpose it is to describe behavior rather than to explain it. At least in theory, biological minimum principles are intended to reflect costs whose minimization is closely related to the animal's

[4] For a comprehensive mathematical treatment of bandit problems, see Berry and Fristedt (1985). An introduction from the perspective of reinforcement learning is provided by Sutton and Barto (1998, Chap. 2).

success at proliferating its genes, i.e., to its *fitness*. If experimental results are consistent with such a minimum principle, this is not only viewed as a confirmation of the predictive power of the minimum principle but also as a confirmation of the effectiveness of natural selection. A biological minimum principle may therefore be viewed not only as a mathematical description of an animal's behavior but also as an *explanation* of that behavior in terms of evolutionary concepts.

Engineering

Minimum principles in engineering are not used to predict behavior but to optimize performance. Typical performance measures are the time it takes to carry out a certain process or the energy that is consumed by it. For most applications, global optimization is impossible or at least impractical. The engineer therefore often assumes a certain system design as given and restricts optimization to a few indeterminate control parameters within that system.

Take robot motion planning as an example. Here, the mechanical design of the robot and the properties of the motors that drive it (such as their energy requirements and their maximum torque output) are assumed to be given, and optimization is restricted to the selection of a movement trajectory. In fact, trajectory planning may be further divided into subprocesses such as the planning of an end-effector trajectory and the conversion of that trajectory into a joint-space trajectory. Optimization is then performed at the level of these subprocesses[5].

Motor Control

The use of minimum principles in motor control is influenced by all three of the disciplines discussed above. Like engineers, motor-control theorists often assume a certain system design as given. For instance, a classic theory in motor control postulates that movement planning is achieved by a hierarchy of transformations that successively generates a hand trajectory, a joint-space trajectory, a sequence of joint torques, and a sequence of muscle-activation patterns (Saltzman, 1979). Except for the joint-space trajectory to torque transformation, all transformations in this hierarchy are indeterminate because they are mappings from lower- to higher-dimensional spaces. To explain the resolution of these indeterminacies, motor control researchers often invoke minimum principles. One example is the popular minimum-jerk theory (Hogan, 1984a; Flash & Hogan, 1985), which postulates that hand trajectories are chosen such that the time integral of the squared magnitude of hand jerk (jerk is the third time derivative of position or, equivalently, the time derivative of acceleration) is minimal. This theory predicts unique hand trajectories, but it does not resolve any other indeterminacies in the movement planning hierarchy. To resolve these, additional performance criteria would need to be specified as functions of joint-space and muscle-space coordinates.

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[5] Some landmark papers on the optimal control of robot manipulators are Hollerbach and Suh (1985), Nakamura and Hanafusa (1985), Sahar and Hollerbach (1986), Suh and Hollerbach (1987), Uchiyama, Shimizu, and Hakomori (1985), and Yoshikawa (1984).

Like evolutionary biologists, most motor-control researchers also believe that minimum principles should reflect some sort of biological utility. Often, however, such utility considerations are only introduced *a posteriori*. For instance, Uno *et al.* (1989) experimented with a number of minimum theories (such as minimum energy, minimum torque, and minimum time) before they decided to postulate the minimum-torque-change theory, the minimum-jerk theory's most influential competitor. The choice of this theory was based on its empirical success, and its potential biological utility (the minimization of wear and tear on the musculoskeletal system) was only discussed as an afterthought.

Other motor control researchers have adopted a completely nonbiological perspective. An example is Hogan (1984a), one of the authors of the minimum-jerk theory, who has argued that minimum theories in motor control should not be "presented as the cause of the behavior they describe but rather as a distillation of its essence." This view is completely abstract and mathematical and is reminiscent of the one taken in physics.

MINIMUM PRINCIPLES AND THE RESOLUTION OF INDETERMINACY

In this section, we shall discuss minimum principles in motor control from an engineering perspective, regarding them as tools for resolving indeterminacies that arise in the process of transforming sensory into motor information. We shall focus on the transformation of a movement goal into a kinematic movement plan. Additional indeterminacies arise in the process of transforming a kinematic plan into muscle or motorneuronal activation patterns (Saltszman, 1979). Cost minimization may also play a role at these lower levels of the motor hierarchy (one performance objective may be the minimization of muscle co-activation: Kamon & Gormley, 1968; Pedotti, Crenna, Deat, Frigo, & Massion, 1989), but we shall not consider this in detail.

In robotics, two basic types of schemes are used for generating kinematic movement plans: Cartesian-space planning and joint-space planning[6].

In Cartesian-space planning, the movement goal (i.e., the desired terminal configuration of the hand) is first translated into a hand trajectory, which is subsequently translated into a joint-space trajectory. The first of these transformations is obviously indeterminate because any hand trajectory that yields the desired end configuration may be used, and the second transformation is also indeterminate because the mapping from joint configuration to hand configuration is many-to-one.

Joint-space planning, in contrast to Cartesian-space planning, does not involve the explicit representation of a hand trajectory. Here planning is organized as follows: First, the end configuration of the hand is converted into a terminal joint-angle configuration, and second, a joint-space trajectory from the initial to the terminal joint-angle configuration is generated. Both of these transformations are indeterminate as well.

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[6] A comprehensive introduction to these planning schemes is provided by Craig (1989, Chap. 7), and additional information may be found in Brady (1982). For a discussion from the perspective of motor control, see Hollerbach (1990).

Cartesian-Space Planning

Consider the task of picking up a cup of coffee with your hand and lifting it to your mouth. To avoid spilling, the cup must be held upright while being lifted, and large inertial forces on the cup need to be avoided. To achieve this, it seems advantageous to make the trajectory of the hand the direct object of control and to view the rest of the arm merely as the mechanical means by which the hand is moved (cf., Hollerbach, 1982). This is the general approach taken in Cartesian-space planning. Bizzi and Mussa-Ivaldi (1989), Flash and Hogan (1985), and many others have argued that this planning scheme is generally used in human motor control, even for tasks that do not require the hand trajectory to be explicitly controlled.

Hand trajectory indeterminacy. Whereas complex manipulation tasks may impose severe constraints on the trajectory of the hand, this is generally not the case for simpler tasks such as unconstrained reaching. Any trajectory that transports the hand from its initial to its target position may be used to successfully perform the latter type of task. But if movement planning is based on the Cartesian-space scheme, a particular hand trajectory needs to be selected. To make this selection, one may proceed as follows: First, associate a cost value with each of these trajectories, and second, select the trajectory for which the cost is lowest. Generally, one may formulate a cost functional (a functional is a scalar-valued function whose domain is a function space) of the form

$$C[\mathbf{r}] = \int_{t_0}^{t_f} g(\mathbf{r}(t), \dot{\mathbf{r}}(t), ..., \mathbf{r}^{(n)}(t))\, dt, \tag{6}$$

where g represents some instantaneous cost associated with the hand configuration $\mathbf{r}(t)$ and its time derivatives. We may then use the methods of the calculus of variations to identify the hand trajectory for which $C[\mathbf{r}]$ is minimized.

In motor control, the best-known minimum principle of this type is the minimum-jerk principle, initially proposed by Hogan (1982, 1984a) and further elaborated by Yashin-Flash (1983) and Flash and Hogan (1985). It suggests that hand trajectories are organized such that the integral

$$C[\mathbf{r}] = \int_{t_0}^{t_f} \|\dddot{\mathbf{r}}\|^2\, dt \tag{7}$$

is minimal. The procedure for solving this minimization problem is analogous to the one used for the two physics problems discussed above. First, we apply the Euler–Poisson equation to Eq. (7) to obtain

$$\mathbf{r}^{(6)}(t) = \mathbf{0}. \tag{8}$$

Next, we integrate this equation six times (which yields a quintic polynomial), choose $t_0 = -T$ and $t_f = T$, for mathematical convenience, and impose the boundary conditions

$$\mathbf{r}(-T) = \mathbf{r}_0 - \mathbf{r}_\Delta, \qquad \mathbf{r}(T) = \mathbf{r}_0 + \mathbf{r}_\Delta, \qquad \dot{\mathbf{r}}(\pm T) = \mathbf{0}, \qquad \ddot{\mathbf{r}}(\pm T) = \mathbf{0} \tag{9}$$

for a movement of amplitude $2\mathbf{r}_\Delta$ and duration $2T$. Thence, we obtain the minimum-jerk trajectory equation

$$\mathbf{r}(\tau) = \mathbf{r}_0 + \tfrac{1}{8}\mathbf{r}_\Delta(15\tau - 10\tau^3 + 3\tau^5), \tag{10}$$

with $\tau \equiv t/T \in [-1, 1]$; the corresponding normalized velocity equation is

$$\frac{d}{d\tau}\mathbf{r}(\tau) = \frac{15}{8}\mathbf{r}_\Delta(1 - 2\tau^2 + \tau^4). \tag{11}$$

If movement duration is dictated by task requirements, the above fully specifies the minimum-jerk solution for a point-to-point trajectory. If, on the other hand, movement duration is not given, it is appropriate to consider its selection as part of the optimization problem. However, for the minimum-jerk principle (as well as for most other minimum principles discussed in this paper) the optimal movement duration is infinity. (This can be easily verified by noticing that jerk cost approaches zero as the duration of a minimum-jerk trajectory approaches infinity.) The most straightforward way to avoid such an absurd prediction is to augment the cost function by adding a secondary term that penalizes duration (Hoff, 1994). If the cost function is altered in this way, its minimum is always achieved at some well-defined value of T, and the optimal trajectory is simply the minimum-jerk trajectory for this duration.

Let us now turn to some important properties of minium-jerk trajectories. From Eqs. (10) and (11) it is straightforward to derive the following trajectory characteristics: (1) straight-line hand paths, (2) linear amplitude scaling, (3) endpoint translation invariance, (4) movement duration invariance, and (5) velocity profile symmetry. The first two characteristics result from $\mathbf{r}_\Delta$ multiplying all τ-dependent terms in Eq. (10); the third one follows from the fact that $\mathbf{r}_\Delta$ remains unchanged if a constant is added to the start and endpoints of the trajectory; the fourth one follows from Eqs. (10) and (11) being independent of T; and the fifth one results from the fact that all terms in Eq. (11) are even in τ.

At the time the minimum-jerk theory was proposed, it concisely summarized the majority of the available arm movement data: Morasso (1981) and Flash and Hogan (1985) had found straight hand paths for various horizontal-plane movements, Hollerbach and Flash (1982) and Soechting and Lacquaniti (1981) had reported data suggesting that trajectory shape is unaffected by changes in movement duration, and the studies of Abend *et al.* (1982), Georgopoulos, Kalaska, and Massey (1981), and Morasso (1981) had shown that velocity profiles are bell-shaped and approximately symmetric. Because of its excellent agreement with these experimental observations, the minimum-jerk principle became one of the most influential motor control theories. In several publications, Flash (1990), Hogan (1988), and Hogan and Flash (1987) further popularized the minimum-jerk theory, and a number of researchers have since expanded its scope in various directions.

One important extension of the theory is concerned with feedback-based trajectory modifications. The need for such modifications may arise from several sources.

First, a movement may deviate from its course due to an unexpected external perturbation; second, normal motor output variability (Schmidt, Zelaznik, Hawkins, Frank, & Quinn, 1979) may also lead to such a deviation; and third, the movement target may unexpectedly change its position during movement execution.

Flash and Henis (1991) showed experimentally that, when the location of a target stimulus is unexpectedly changed (either shortly before or shortly after movement onset), the movement is initially directed toward the original target before smoothly changing direction toward the new target location. They suggested that trajectories of this kind can be modeled as the temporal superposition of two minimum-jerk trajectory components, with the first component extending from the initial hand position to the original target location and the second one extending from there to the shifted target location. Results consistent with this superposition scheme were reported by Flash and Henis (1991) and Henis and Flash (1992, 1995), and further support for it comes from infant reaching movements, whose multipeaked velocity profiles are well modeled as temporal superpositions of minimum-jerk trajectory units (Berthier, 1996).

While the superposition scheme uses feedback only at isolated points in time (e.g., when a target switch occurs), it is also conceivable that feedback is used on a more continuous basis. This can by achieved by an optimal feedback control design that uses an estimate of the arm's current state and an estimate of target location as input and generates a jerk control signal as output. Hoff and Arbib (1993) discuss this control scheme in detail and show that it is consistent with results from various target perturbation studies.

A second direction in which the minimum-jerk theory has been extended involves the consideration of tasks with trajectory constraints. One relatively simple constraint arises when the movement is required to pass through a via point. In this case, the minimum-jerk trajectory can be shown to be composed of two trajectory components, each of which is a fifth-order polynomial (Flash & Hogan, 1985). The first component starts at rest (zero velocity and acceleration) and extends to the via point, while the second component starts at the via point and terminates with zero velocity and acceleration at the target location. At the via point, both components must have the same velocities and accelerations; but these do not need to be zero. Instead, these two values, as well as the time of passage through the via point, are additional parameters that must be optimized. The minimum-jerk model's predictions for via-point movements were tested by Flash and Hogan (1985), who found good quantitative agreement between theoretical and experimental trajectories. Of particular interest was the observation that the minimum-jerk trajectories predicted roughly equal durations for the two trajectory components, even if these differed considerably in length. This prediction, which is closely related to the *isochrony principle* (Viviani & Terzuolo, 1982), was found to be consistent with the experimental data.

Usually, in a via-point movement, only the position at the via point is given, while the velocity is determined as part of the optimization problem. In other applications, it may, however, be useful to dictate the velocity at the via point as well. Edelman and Flash (1987) showed that, by doing so, a variety of trajectory shapes can be generated, and these may be used to simulate strokes that closely resemble those found in handwriting. Furthermore, if one has two strokes whose boundary conditions are such that the final velocity of the first matches the initial velocity of the second, the strokes may be joined to produce a more complex trajectory; a third stroke may then be concatenated to this trajectory, and so forth.

A particular instance of a complex trajectory arises when the start and endpoints of a chain of concatenated trajectory components coincide. In that case, one may remove the constraint that the initial and final velocities and accelerations must be zero, and the closed trajectory may be traversed repeatedly, resulting in a cyclical movement. Viviani and Flash (1995) had participants perform various cyclical figure tracing tasks and showed that a concatenation of minimum-jerk trajectory components can be used to successfully predict the experimental trajectories. These trajectories were furthermore shown to be consistent with the *two-thirds power law* (Lacquaniti, Terzuolo, & Viviani, 1983; Viviani & Schneider, 1991), which suggests a close relationship between this law and the minimum-jerk principle.

The above discussion gives an impression of the wide variety of tasks to which the minimum-jerk theory has been successfully applied. However, despite these successes, the theory has not gone unchallenged. For instance, there is now ample evidence that, contrary to the theory's prediction, hand paths of point-to-point movements are not shape-invariant throughout the workspace. In numerous experiments, it has been shown that horizontal-plane point-to-point movements are essentially straight if they are performed near the workspace center and are directed along the anterior–posterior axis (Flash & Hogan, 1984; Haggard, Hutchinson, & Stein, 1995; Miall and Haggard, 1995; Morasso, 1981), but if they start or end near the workspace boundaries, they are noticeably curved (Osu, Uno, Koike, & Kawato, 1997; Suzuki, Yamazaki, Mizuno, & Matsunami, 1997; Uno *et al.*, 1989). Curved hand paths have also been found by Atkeson and Hollerbach (1985) and Flanagan and Ostry (1990) for vertical as well as for inward-and-upward diagonal movements in the sagittal plane, by Lacquaniti, Soechting, and Terzuolo (1986) for inward-and-upward movements directed toward the subjects' noses, and by Desmurget *et al.* (1995) for outward-and-upward reach-to-grasp movements. Other inconsistencies between experimentally observed hand paths and minimum-jerk predictions have been found for vertical-plane movements through a via point (Furuna & Nagasaki, 1993) and for horizontal-plane obstacle avoidance movements (Dean & Brüwer, 1994).

Certain details of the temporal characteristics of arm movements also seem to be inconsistent with the minimum-jerk theory. In studies of one-degree-of-freedom horizontal-plane forearm movements, Nagasaki (1989) and Wiegner and Wierzbicka (1992) found that the ratio of peak to average velocity is a function of movement duration, which indicates that velocity profiles are not shape invariant under changes in movement duration. Furthermore, it was found that velocity profiles are not always symmetric; they tend to be right-skewed for slow movements (Moore & Marteniuk, 1986; Nagasaki, 1989) and left-skewed for fast movements (Wiegner & Wierzbicka, 1992).

Some of the minimum-jerk model's controversial predictions such as straight hand paths (for point-to-point movements) and velocity profile symmetry extend to other Cartesian-space cost measures such as squared acceleration and squared snap (Yashin-Flash, 1983) and, more generally, to any cost functional of the form

$$\int_{t_0}^{t_f} \|\mathbf{r}^{(n)}(t)\|^2 \, dt. \tag{12}$$

This expression comprises a rather broad class of cost functionals, but it certainly does not exhaust all possible formulations of a Cartesian-space minimum principle; there may be others that make entirely different predictions.

Inverse kinematics indeterminacy. Most of the original work on minimum principles was done in the context of planar movements involving two mechanical degrees of freedom (DOF), one at the shoulder and one at the elbow. For these movements, the transformation from hand-space to joint-space coordinates (usually referred to as the *inverse kinematics* transformation) is completely determinate, i.e., the trajectory of the hand uniquely determines the joint-space trajectory of the arm, given the biomechanical joint-range limits are taken into account. Only recently have researchers begun to given more attention to unrestricted movements in all three spatial dimensions (e.g., Soechting, Buneo, Herrmann, & Flanders, 1995; Soechting & Flanders, 1992). For these, the inverse kinematics transformation is indeterminate.

Unrestricted movement of the hand is effected by the rotation of two major joints, the three-DOF spherical shoulder joint and the one-DOF elbow joint.[7] We shall use the symbol $\theta = (\theta_1, \theta_2, \theta_3, \theta_4)^{\mathrm{T}}$ to denote the joint-angle configuration of the arm and $\boldsymbol{\Theta}^4$ to denote the set of all biomechanically admissible joint-angle configurations. The shoulder joint angles θ_1, θ_2, and θ_3 specify the jaw, pitch, and roll of the upper arm, respectively, and the elbow joint angle θ_4 specifies the interior angle between the upper arm and forearm. This description of arm configuration is, of course, not unique. It is a convenient choice for the discussion that follows, but other coordinate systems may very well be more meaningful from a psychophysical perspective (Soechting & Ross, 1984).

Independent of the choice of coordinates, it should be clear that the shoulder and the elbow joint have a total of four mechanical DOF, one in excess of the three DOF necessary to uniquely specify a location in Cartesian space. This excess DOF makes it possible to rotate the elbow around an axis through the hand and the shoulder, while maintaining the hand in a fixed position (see Fig. 1).

The curve that is traced by the rotating elbow may be found as the intersection of two spheres, one with center at the shoulder and radius l_1 and the other one with center at the hand and radius l_2, where l_1 and l_2 are the lengths of the upper arm and forearm, respectively (cf., Hollerbach, 1985). The following result gives an analytical expression for this curve:

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[7] Small changes in hand position may also be effected by the radioulnar (one DOF) and radiocarpal (two DOF) articulations of the forearm and wrist. For simplicity, these are disregarded (i.e., the wrist is assumed to be braced). Furthermore, only the position of the hand is considered, but its orientation is ignored.

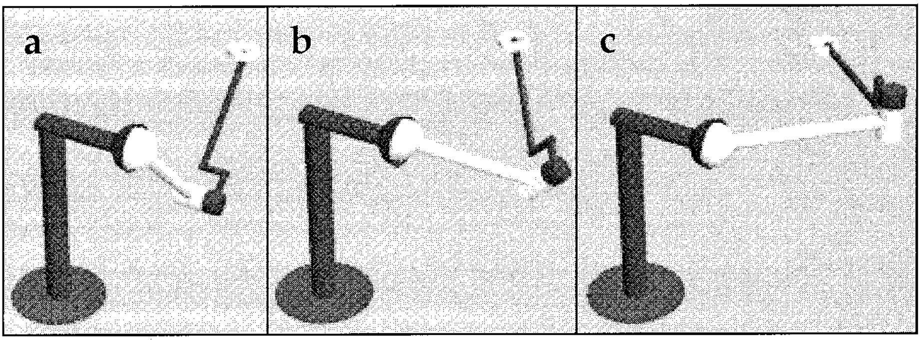

FIG. 1. The inverse kinematics indeterminacy for an arm with three-DOF spherical shoulder joint and one-DOF elbow joint. The figure depicts three different arm configurations that result in identical hand positions. The configurations shown in (b) and (c) are obtained from the one shown in (a) by rotation of the elbow around an axis through the shoulder and the hand.

LEMMA. *Consider an arm with three-DOF spherical shoulder joint and one-DOF elbow joint. If the hand is positioned at* $\mathbf{r}$, *the elbow position* $\mathbf{q}$ *must be an element of the set* $\mathbf{Q}(\mathbf{r}) = \{\mathbf{q} \mid \mathbf{q} = \mathbf{q}(\mathbf{r}, \gamma), \gamma \in [-\pi, \pi)\}$, *with*

$$\mathbf{q}(\mathbf{r}, \gamma) = \frac{1}{2r_\varphi} \mathbf{R}_z(r_\theta)\, \mathbf{R}_y(r_\phi) \left(\begin{pmatrix} r_\varphi^2 + l_1^2 + l_2^2 \\ 0 \\ 0 \end{pmatrix} + \sqrt{4 l_1^2 r_\varphi^2 - (r_\varphi^2 + l_1^2 - l_2^2)^2} \begin{pmatrix} 0 \\ \cos\gamma \\ \sin\gamma \end{pmatrix} \right),$$

where $\mathbf{R}_z$ *and* $\mathbf{R}_y$ *are matrix operators that perform z-axis and y-axis rotations, respectively,* $r_\theta \equiv \tan^{-1}(r_1, r_2)$, $r_\phi \equiv \sin^{-1}(r_3/r_\varphi)$, *and* $r_\varphi \equiv \|\mathbf{r}\|$ *are spherical coordinates, and* $\tan^{-1}(x, y) \equiv \tan^{-1}(y/x) + \mathrm{sgn}(y)(1 - \mathrm{sgn}(x))\, \pi/2$.

The above lemma illustrates that any hand position $\mathbf{r}$ is compatible with an infinite number of elbow positions $\mathbf{q} \in \mathbf{Q}(\mathbf{r})$ and hence with an infinite number of joint-angle configurations. A unique joint-angle configuration is, however, defined if both the hand position $\mathbf{r}$ and the elbow position $\mathbf{q}$ are given.[8] The set of all joint-angle configurations that can place the hand at $\mathbf{r}$ may therefore be generated by moving the elbow along the curve $\mathbf{q}(\mathbf{r}, \gamma)$, while the hand rests at $\mathbf{r}$. An explicit expression for this inverse kinematics set is given by the following theorem:

THEOREM (Inverse kinematics). *All inverse kinematics solutions of an arm with three-DOF spherical shoulder joint and one-DOF elbow joint are members of the set* $\mathbf{\Phi}(\mathbf{r}) = \{\theta(\mathbf{r}, \gamma) \mid \gamma \in \Gamma(\mathbf{r})\}$, *with*

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[8] Strictly, there are always two joint-angle configurations, θ and θ', that can generate identical position pairs $\{\mathbf{q}, \mathbf{r}\}$. However, given the limited biomechanical range of motion of the elbow joint, i.e., $\theta_4 \subseteq [0, \pi)$, only one of the two is physically realizable (see Engelbrecht, 1997, for further detail). For all practical purposes, we may therefore treat the relation between $\{\mathbf{q}, \mathbf{r}\}$ and θ as one-to-one.

$$\begin{aligned}
&\theta_1(\mathbf{r}, \gamma) = \tan^{-1}(q_2(\mathbf{r}, \gamma)/q_1(\mathbf{r}, \gamma)),\\
&\theta_2(\mathbf{r}, \gamma) = \tan^{-1}(\sqrt{q_1^2(\mathbf{r}, \gamma) + q_2^2(\mathbf{r}, \gamma)}, q_3(\mathbf{r}, \gamma)),\\
&\theta_3(\mathbf{r}, \gamma) = \tan^{-1}(2l_1(r_2 q_1(\mathbf{r}, \gamma) - r_1 q_2(\mathbf{r}, \gamma)), 2r_2 l_1^2 - (r_\varphi^2 + l_1^2 + l_2^2)\, q_3(\mathbf{r}, \gamma)),\\
&\theta_4(\mathbf{r}, \gamma) = \cos^{-1}((r_\varphi^2 + l_1^2 - l_2^2)/2l_1 l_2),\\
&\quad \Gamma(\mathbf{r}) = \{\gamma \in [-\pi, \pi) \mid \theta(\mathbf{r}, \gamma) \in \boldsymbol{\Theta}^4\},
\end{aligned}$$

where $\mathbf{q}(\mathbf{r}, \gamma)$ *is as defined in the above lemma.*

For brevity, we omit a proof of this result and refer the reader to Engelbrecht (1997).

Besides confirming that the inverse kinematics transformation is indeed indeterminate, the above theorem provides an explicit parametrization of the set of all inverse kinematics solutions by means of the function $\theta(\mathbf{r}, \gamma)$. Some significant benefits are associated with this. Consider, for instance, the problem of identifying a joint-space trajectory $\theta(t)$ that minimizes some cost function while transporting the hand along a trajectory $\mathbf{r}(t)$. Without the above result, this would be represented as a variational problem involving four unknown functions (for the four DOF of the arm) and three constraint equations (resulting from the requirement that the hand must follow a given trajectory). The above theorem allows us to reduce the problem to an unconstrained one that involves only the single scalar-valued function $\gamma(t)$; this greatly simplifies the treatment of such problems, and we shall take advantage of it on several occasions.

From the above result, we know that each pair of functions $\{\mathbf{r}(t), \gamma(t)\}$ uniquely determines a joint-space trajectory. Consequently, if the hand trajectory $\mathbf{r}(t)$ is prescribed, all that needs to be determined is the function $\gamma(t) \in \Gamma(\mathbf{r}(t))$. Clearly, there is an infinite number of such functions. To identify a particular one, we may define a functional of the form

$$C[\gamma] = \int_{t_0}^{t_f} g(\gamma(t), \dot{\gamma}(t), \ldots, \gamma^{(n)}(t))\, dt \tag{13}$$

that assigns a different cost to each candidate function. A unique joint-space trajectory may then be identified by requiring that $\gamma(t)$ be chosen to minimize $C[\gamma]$. For instance, one could consider applying the minimum-jerk principle to the inverse kinematics transformation by defining jerk-cost in joint-space coordinates:

$$g(t) = \|\ddot{\theta}(\mathbf{r}(t), \gamma(t), \dot{\gamma}(t), \ddot{\gamma}, \dddot{\gamma}(t))\|^2. \tag{14}$$

Possibly as a result of the research focus on nonredundant two-DOF planar movements, this has not been attempted, nor have motor control researches applied any other minimum principle to this trajectory conversion problem[9]. The inverse

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[9] There is, however, an abundance of related robotics work. For a comprehensive review, see Nakamura (1991).

kinematics transformation has nevertheless attracted considerable research, albeit in a different context, that of joint-space planning.

Joint-Space Planning

As reviewed by Craig (1989, Chap. 7), Cartesian-space planning is associated with a number of problems that may be avoided if movements are directly planned in joint-space coordinates. All of these problems result from the nonlinear relation between hand-space and joint-space coordinates. We shall illustrate one of these by considering simple two-DOF planar arm movements. For these, it is relatively straightforward to show that the ratio of the magnitude of joint velocity to the magnitude of hand velocity obeys

$$\frac{\|\dot{\theta}\|}{\|\dot{\mathbf{r}}\|}=\frac{\sqrt{l_2^2(\dot{r}_1\,\mathrm{c}\phi+\dot{r}_2\,\mathrm{s}\phi)^2+(l_1\dot{r}_1\,\mathrm{c}\theta_1+l_2\dot{r}_1\,\mathrm{c}\phi+l_1\dot{r}_2\,\mathrm{s}\theta_1+l_2\dot{r}_2\,\mathrm{s}\phi)^2}}{l_1 l_2\sqrt{\dot{r}_1^2+\dot{r}_2^2}\,\mathrm{s}\theta_2} \tag{15}$$

$$\mathrm{c}\alpha\equiv\cos\alpha,\qquad \mathrm{s}\alpha\equiv\sin\alpha,\qquad \phi\equiv\theta_1+\theta_2,$$

where l_1 and l_2 are the lengths of the upper arm and forearm, respectively, $\mathbf{r}\equiv(r_1, r_2)^{\mathrm{T}}$ are the coordinates of the hand, and $\theta\equiv(\theta_1, \theta_2)^{\mathrm{T}}$ are the joint angles at the shoulder and the elbow. As can be seen from the above expression, the ratio $\|\dot{\theta}\|/\|\dot{\mathbf{r}}\|$ varies greatly as a function of joint-angle configuration, and it, in fact, diverges if the arm approaches the singular configurations $\theta=(\theta_1, 0)^{\mathrm{T}}$ or $\theta=(\theta_1, \pi)^{\mathrm{T}}$. Hand trajectories that bring the arm into the vicinity of these singularities may therefore require extremely inefficient or even unrealizably large joint velocities (for an example, see Fig. 2) and should thus be avoided whenever

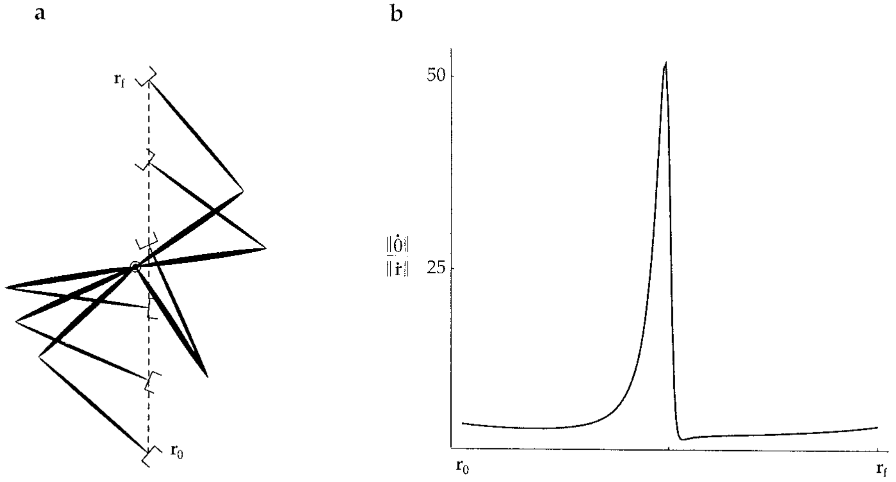

FIG. 2. Straight-line hand movement. Movement starts at $\mathbf{r}_0$ and terminates at $\mathbf{r}_f$. Near the middle of the path, the shoulder is forced to perform a full π rad rotation that brings the hand only minimally closer to the target (Panel a). At that point, the ratio of the magnitude of joint velocity to the magnitude of hand velocity (in units of rad/m) increases dramatically (Panel b). From an engineering point of view, hand paths of his kind are extremely inefficient and should be avoided whenever possible.

possible. But it is difficult to see how this can be done if hand trajectory planning strictly precedes the generation of a joint-space trajectory. The most straightforward way of avoiding these singularities is to plan a movement directly in joint-space coordinates. From an engineering point of view, joint-space planning is therefore preferable over Cartesian-space planning whenever the task does not explicitly constraint the trajectory of the hand.

End-configuration indeterminacy. The first step in joint-space planning involves the computation of the inverse kinematics transformation, which has already been discussed in the context of Cartesian-space planning. But whereas in Cartesian-space planning this transformation is computed along a complete hand trajectory, joint-space planning invokes its computation only at the movement endpoint $\mathbf{r}_f$. As a consequence, we may now resolve the inverse kinematics indeterminacy by simply requiring that some static cost

$$C(\gamma) = g(\theta(\mathbf{r}_f, \gamma)) \tag{16}$$

be minimized. Note that this is not a problem in the calculus of variations but is instead a straightforward parameter optimization problem; its solution may simply be determined from the following two conditions:

$$\frac{d}{d\gamma} C(\gamma) = 0, \qquad \frac{d^2}{d\gamma^2} C(\gamma) > 0. \tag{17}$$

A variety of end-configuration costs may be considered, ranging from engineering measures such as manipulability (Yoshikawa, 1985) to psychological measures such as postural comfort. Concerning the latter, there are psychophysical data indicating that subjects consistently rate certain postures (joint-angle configurations) as more comfortable than others (Cruse, 1986; Cruse, Wischmeyer, Brüwer, Brockfeld, & Dress, 1990; Rosenbaum, Vaughan, Jorgensen, Barnes, & Stewart, 1993). Cruse (1986) argued that each joint has an associated discomfort function and that these functions can be added to obtain a measure of overall postural discomfort. The discomfort associated with an individual joint is highest near the joint's biomechanical range limits and lowest for some optimal configuration, which tends to be near the middle of the joint's range of motion.

To test the hypothesis that target joint-angle configurations are chosen to minimize posture discomfort, Cruse *et al.* (1990) asked subjects to perform a number of horizontal-plane reaching movements involving the shoulder, the elbow, and the wrist. Additionally, subjects were asked to rate the discomfort they associated with a variety of different arm configurations. The results of these kinematic and psychophysical measurements were found to be consistent with the minimum-discomfort hypothesis.

Additional support for this hypothesis comes from a number of studies by Rosenbaum and his collaborators (Rosenbaum & Jorgensen, 1992; Rosenbaum, Vaughan, Marchak, Barnes, & Slotta, 1990; Rosenbaum, Vaughan, Barnes, & Jorgensen, 1992; Rosenbaum *et al.*, 1993). The general type of task investigated in these studies required a subject to grasp an object and then transport it to a new location or to place it in a new orientation. The final position and orientation of the object were prespecified, but the subject was free to decide how to grasp the object (e.g., by using an underhand or an overhand grip). In correspondence with the results obtained by Cruse *et al.* (1990), it was found that subjects consistently chose grasps that minimized the awkwardness of final wrist orientation.

A problem with the above studies is that the concept of discomfort is not well defined. For instance, by rating one posture as being less uncomfortable than another, a person may simply mean that, given the choice, he or she would prefer the former posture over the latter one. If discomfort is interpreted in this fashion, the minimum-discomfort hypothesis translates into the following circular statement: “People prefer to adopt those postures that they prefer to adopt.” Of course, discomfort may also be taken to refer to some more objective cost measure such as proximity to the biomechanical joint-range limits or magnitude of the torques necessary to maintain a given posture. However, in each of these case, it seems advantageous to replace the term discomfort with the more specific physical interpretation.

In general, two types of cost measures may be differentiated: kinetic costs and kinematic costs. Kinetic costs derive from the muscle-generated forces or torques that are applied to the arm. Kinematic costs, on the other hand, are independent of these and derive exclusively from the configuration of the arm and its time derivatives. An external force that is applied to the arm may therefore affect the former type of cost, but it cannot affect the latter one. Cruse *et al.* (1990) carried out an experiment that speaks to this distinction. In this experiment, target-directed arm movements were performed under two conditions; in one condition, a spring was mounted across the subject’s elbow, while in the other one, no external force was applied to the arm. For all investigated movements, the terminal joint-angle configurations were significantly different in the two conditions. This finding is inconsistent with any exclusively kinematics-based minimum theory; it can only be explained if kinetic costs are also taken into consideration.

A kinetic cost of particular interest is the sum of the squared torques that are applied to the joints during movement or while a posture is maintained. In robot manipulators that are driven by electric torque motors, this measure is directly proportional to the power consumption of the motors. More importantly, this measure also appears to be approximately proportional to the metabolic energy that is consumed by the muscles when they generate torque at a joint (Hogan, 1984b).

The relation between arm movement kinematics and torque can be derived from the Lagrange equation

$$\mathbf{N}(\theta, \dot{\theta}, (\ddot{\theta}) = \frac{d}{dt} \frac{\partial L(\theta, \dot{\theta})}{\partial (\ddot{\theta}} - \frac{\partial L(\theta, \dot{\theta})}{\partial \theta}, \tag{18}$$

where $\mathbf{N}$ denotes torque, and L is defined as the difference between the kinetic and potential energies of the arm; i.e., $L(\theta, \dot{\theta}) \equiv T(\theta, \dot{\theta}) - U(\theta)$. If we consider only the torques necessary to maintain the terminal arm configuration $\theta(\mathbf{r}_f, \gamma)$, Eq. (18) reduces to the much simpler expression

$$\mathbf{N}_f(\theta) = \frac{\partial U(\theta(\mathbf{r}_f, \gamma))}{\partial \theta} \tag{19}$$

because the kinetic energy is zero when the arm is at rest. Since $\mathbf{N}_f$ is uniquely determined by $\theta(\mathbf{r}_f, \gamma)$, the squared posture-maintenance torque at the movement endpoint may be written as

$$C(\gamma) = \|\mathbf{N}_f(\theta(\mathbf{r}_f, \gamma))\|^2, \tag{20}$$

from which the minimum-torque configuration $\theta(\mathbf{r}_f, \gamma^*)$ may be found by application of Eq. (17).

In a number of experiments, Engelbrecht (1997) tested the hypothesis that final joint-angle configurations are chosen to minimize squared posture-maintenance torque (SPMT) as defined by Eq. (20). The general type of task investigated in these experiments had three components: (1) a target-directed arm movement, (2) a period of posture maintenance at the target, usually lasting between 15 and 45 s, and (3) a movement returning the arm to its initial posture. The studied movements involved all three rotational DOF at the shoulder and one DOF at the elbow; as a consequence, each target location could be reached with an infinite number of joint-angle configurations (see above theorem). When subjects performed the experimental task while holding a 2.27-kg weight, their target joint-angle configurations were well predicted by the minimum-SPMT theory (see Fig. 3), but when the same task was performed without a weight, the theory failed. In the latter experiment, target joint-angle configurations varied greatly as a function of the arm posture at movement onset (see also Fischer, Rosenbaum, & Vaughan, 1997; Soechting *et al.*, 1995), which suggests that static configuration costs, whether kinematic or kinetic, are generally not sufficient to predict terminal joint-angle configurations. Costs arising from the transition between initial configuration and terminal configuration must also be taken into consideration.

Trajectory indeterminacy. From Engelbrecht's (1997) experimental results, one may conclude that the selection of a target joint-angle configuration and the selection of a joint-space trajectory are problems that need to be resolved simultaneously. For this purpose, it is useful to formulate a cost function of the form

$$C[\theta] = \int_{t_0}^{t_f} g_1(\theta(t), \dot{\theta}(t), ..., \theta^{(n)}(t))\, dt + g_2(\theta(t_f)), \tag{21}$$

where g_1 is some instantaneous cost associated with $\theta(t)$ and its time derivatives, and g_2 is some static cost associated with the terminal joint-angle configuration $\theta(t_f) = \theta(\mathbf{r}_f, \gamma)$. A unique joint-space trajectory (including the terminal configuration $\theta(t_f)$) may then be identified by requiring that it be chosen to minimize the above

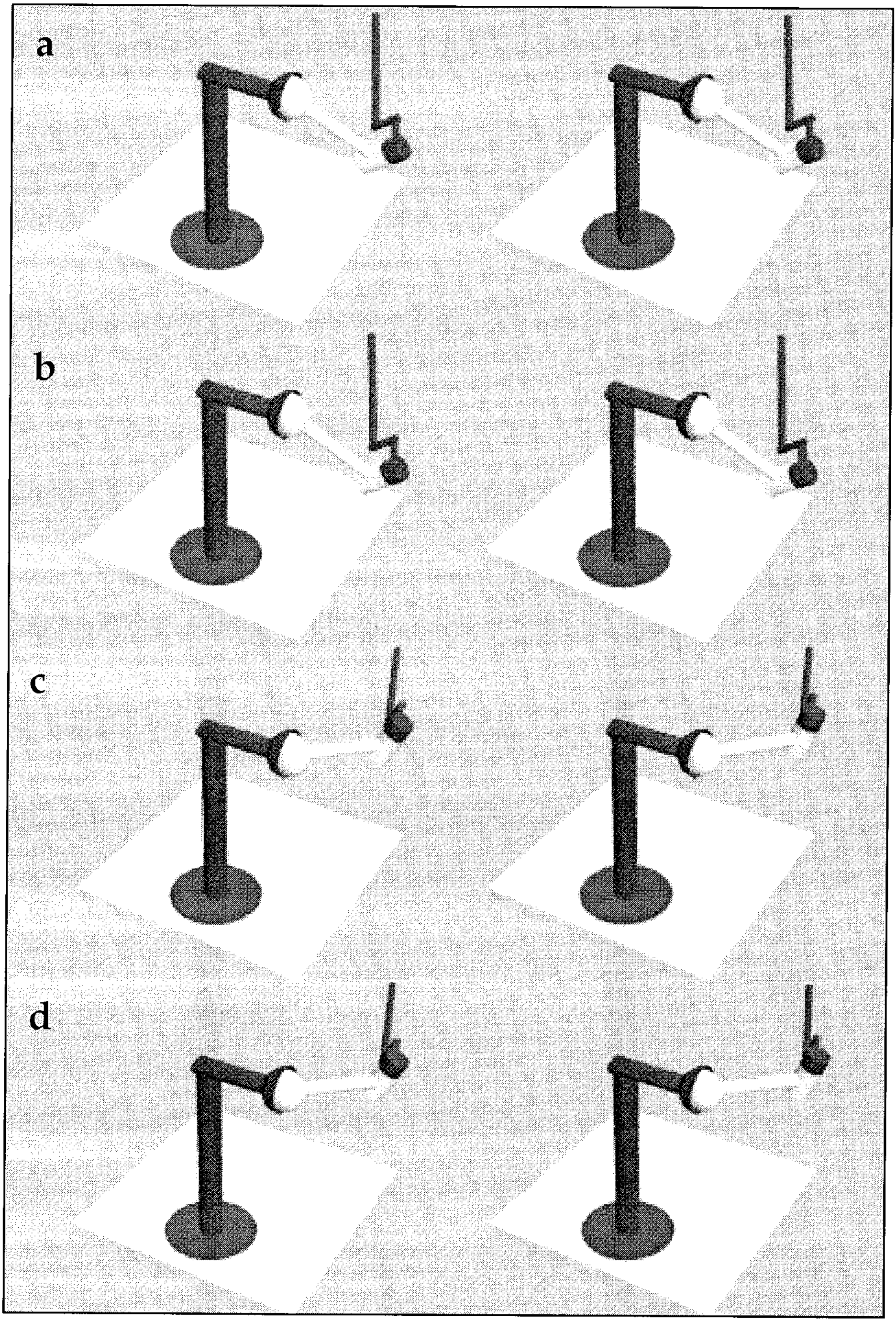

FIG. 3. Observed vs. predicted performance (minimum-SPMT criterion). Movement was either directed toward a Near target or a Far target, and movement started with the elbow in either a High position to the right of the shoulder or a Low position near the hip. The subjects held a 2.27-kg weight in their right hands and were required to maintain the final posture for 30 s. Mean final postures (averaged over four subjects and sic repetitions) for the (a) Near-High, (b) Near-Low, (c) Far-High, and (d) Far-Low conditions are shown in the left column, and the corresponding minimum-SPMT postures are shown in the right column.

cost function subject to a set of appropriate boundary conditions. At movement onset, $\theta(t_0) = \theta_0$ must be satisfied since the initial arm configuration is fixed. At movement termination, on the other hand, we do not need to impose such a rigid constraint, but may instead allow any arm configuration $\theta(t_f) \in \mathbf{\Phi}(\mathbf{r}(t_f))$ that positions the hand at the target. With this more relaxed constraint, the selection of the end-configuration becomes part of the optimization problem, and the end-configuration and joint-space-trajectory indeterminacies are resolved simultaneously.

Uno *et al.*'s (1989) minimum-torque-change principle is the best-known minimum theory of this form. It has inspired a wealth of experimental and theoretical work (Kawato, 1992, 1996; Kawato, Maeda, Uno, & Suzuki, 1990; Osu *et al.*, 1997; Wada & Kawato, 1993, 1995; Wada, Koike, Vatikiotis-Bateson, & Kawato, 1995), and it is worth considering in more detail. As a first step, let us examine how torque change depends on the joint-space trajectory of the arm. This relation may be derived from the Lagrange Eq. (18); hiding the details, it takes the form

$$\dot{\mathbf{N}}(t) = \mathbf{D}(\theta(t), \dot{\theta}(t), (\ddot{\theta}(t), \dddot{\theta}(t)), \tag{22}$$

where $\mathbf{D}$ represents a nonlinear third-order vector differential equation. The torque-change cost is therefore a function of $\theta(t)$ and its first three time derivatives and is given by the equation

$$C = \int_{t_0}^{t_f} \|\dot{\mathbf{N}}(\theta(t), \dot{\theta}(t), (\ddot{\theta}(t), \dddot{\theta}(t))\|^2 \, dt, \tag{23}$$

which does not include an end-configuration cost term since $\dot{\mathbf{N}}(t) = \mathbf{0}$ when the arm is at rest.

To find the minimum-torque-change trajectory, we now apply the Euler–Poisson equation to Eq. (23). This yields

$$\frac{\partial \|\dot{\mathbf{N}}\|^2}{\partial \theta} - \frac{d}{dt} \frac{\partial \|\dot{\mathbf{N}}\|^2}{\partial \dot{\theta}} + \frac{d^2}{dt^2} \frac{\partial \|\dot{\mathbf{N}}\|^2}{\partial \ddot{\theta}} - \frac{d^3}{dt^3} \frac{\partial \|\dot{\mathbf{N}}\|^2}{\partial \dddot{\theta}} = \mathbf{0}, \tag{24}$$

a nonlinear sixth-order vector differential equation. It is generally not possible to solve this equation analytically, but numerical methods may be used to find individual solutions for given sets of boundary conditions and limb-segment parameters (inertias, masses, and viscosities). Uno *et al.* (1989) did this for several two-DOF horizontal-plane movements and obtained trajectories that were highly consistent with experimentally observed ones. In particular, they found that the hand paths of horizontal-plane arm movements (which are essentially straight for inward and outward movements near the workspace center but are noticeably curved for transverse movements) were well predicted by the minimum-torque-change theory. This represented an important improvement over the minimum-jerk theory, which does not predict these workspace variations in hand-path curvature: Minimum-jerk hand paths are straight regardless of workspace location and movement direction, unless additional assumptions, such as incomplete control of movement execution (Flash, 1987; but see Gomi & Kawato, 1996; Osu *et al.*, 1997) or visual misperception of hand-path curvature (Wolpert, Ghahramani, & Jordan, 1994; but see Osu *et al.*, 1997), are introduced.

While Uno *et al.* (1989) obtained their results by use of numerical methods, Engelbrecht and Fernández (1997) showed that for the important special case of one-DOF planar-horizontal movements, minimum-torque-change trajectories may also be found analytically. For these movements, it can be shown that minimum-torque-change trajectories exhibit a number of properties that hold for any movement duration and for any set of arm-dynamical parameters. These properties are: (1) a symmetric velocity profile, (2) a ratio of peak velocity over average velocity greater than 3/2 but smaller than 15/8, and (3) an acceleration peak after less than 21.13% of the total movement duration. Although these properties may hold for certain types of arm movements, a review of the experimental literature shows that they are not universally correct: As was mentioned above, there is evidence that the velocity profiles of slow movements are generally right-skewed (Moore & Marteniuk, 1986; Nagasaki, 1989), while those of extremely fast movements are left-skewed (Wiegner & Wierzbicka, 1992). Fast movements also tend to have ratios of peak to average velocity that significantly exceed 15/8 (Nagasaki, 1989; Wiegner & Wierzbicka, 1992), and their acceleration peaks may occur significantly later than 21.13% of total movement duration (Baba & Marteniuk, 1983).

Summary

None of the considered minimum principles is fully accurate. The minimum-jerk principle, as well as all other minimum principles that are instances of Eq. (12), incorrectly predicts straight hand paths regardless of workspace location. The minimum-discomfort and minimum-SPMT theories only predict terminal joint angle configurations but not trajectories. These theories make accurate predictions under certain circumstances. In general, however, they fail because they cannot account for the dependency of terminal arm posture on initial arm posture. The minimum-torque-change principle stands out from the others because it correctly predicts the workspace-dependency of hand-path curvature, but its predictions concerning the temporal characteristics of movement are not entirely accurate.

DEVELOPMENT AND TESTING OF MINIMUM THEORIES

Without proper constraints, the development of a minimum theory may easily become an exercise in circular reasoning. Gould and Lewontin (1979) argued that in evolutionary biology such circular reasoning is not uncommon, caricaturing it as follows: Convinced of the omnipotence of natural selection, the evolutionary biologist studies the behavior of an animal with the intention to identify a performance measure with respect to which the behavior is optimal. Given sufficient imagination, the evolutionary biologist always succeeds at finding such a measure, which is then presented as evidence that animal behavior is indeed optimally adapted.

Although others (Maynard Smith, 1978; Mayr, 1983) have disagreed with Gould and Lewontin (1979) about the extent to which such circular arguments are actually employed in evolutionary biology, there seems to be some consensus that minimum (or optimality) principles lend themselves to such arguments if proper care is not taken.

Surprisingly, no such concerns have been voiced with respect to minimum principles in motor control. Yet, as we shall see below, the same danger of circular reasoning exists for these as well.

A Posteriori Theory Development

It is not particularly difficult to develop a minimum theory that makes reasonable predictions. The following example will illustrate that. As an alternative to the minimum-torque-change principle, consider a minimum theory that associates costs with both the first and the second time derivative of torque, so that

$$C[\theta] = \int_{t_0}^{t_f} (\ddot{N}^2(t) + \lambda^2 \dot{N}^2(t))\, dt. \tag{25}$$

The relative weight of the terms $\dot{N}^2$ and $\ddot{N}^2$ is determined by λ, which will be given in units of kHz. For one-DOF horizontal-plane movements, it is straightforward to find the trajectory that minimizes this cost functional. If viscosity is negligible, we have

$$N(t) = I\ddot{\theta}, \tag{26}$$

where I is the inertia of the limb segment. We insert Eq. (26) into Eq. (25) and apply the Euler–Poisson equation to obtain the homogeneous second-order ordinary differential equation

$$-\lambda^2 \phi = \ddot{\phi}, \qquad \phi \equiv \theta^{(6)} \tag{27}$$

with solution

$$\phi(t) = a_6 \lambda^6 \cosh \lambda t + a_7 \lambda^6 \sinh \lambda t; \tag{28}$$

sixfold integration then yields

$$\theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 \cosh \lambda t + a_7 \sinh \lambda t. \tag{29}$$

The unknown coefficients of this equation can be found from the boundary conditions. For mathematical convenience, we choose the origin of the time axis such that $t_0 = -T$ and $t_f = T$. For the same reason, and without loss of generality—the cost function (25) does not depend on $\theta(t)$ and is therefore translation invariant—we choose $\theta(-T) = -\theta_f$ and $\theta(T) = \theta_f$. With movement starting and ending at rest, and assuming continuity of $\theta(t)$ and its first three time derivatives, we then have

$$\theta(-T)=-\theta_f, \qquad \theta(T)=\theta_f, \qquad \dot{\theta}(\pm T)=\ddot{\theta}(\pm T)=\dddot{\theta}(\pm T)=0. \tag{30}$$

From Eq. (30) we determine the coefficients $a_0 - a_7$, and inserting these into (29), we obtain

$$\theta(\tau)=\theta_f Q(\omega)\left(\frac{15}{8}(8\omega+\omega^3-5\omega^2\tanh\omega)\,\tau+\frac{5}{4}(3\omega^2\tanh\omega-\omega^3)\,\tau^3\right.$$
$$\left.+\frac{3}{8}(\omega^3-\omega^2\tanh\omega)\,\tau^5-15\,\frac{\sinh\omega\tau}{\cosh\omega}\right), \tag{31}$$

with $\tau\equiv t/T\in[-1,1]$, $\omega\equiv\lambda T$, and $Q(\omega)\equiv 1/(15\omega+\omega^3-(15+6\omega^2)\tanh\omega)$.

Equation (31) represents a family of trajectories that is parametrized by ω. To obtain a few example trajectories, we let $\lambda=0.1$ kHz and consider the following two movement durations: $2T=100$ ms and $2T=1000$ ms. The time-normalized trajectories that correspond to these values are shown in Fig. 4 (dashed curves), together with a minimum-torque-change trajectory[10] (solid curve). As can be seen from the figure, trajectories that obey Eq. (31) are duration dependent: The 100-ms movement is characterized by a velocity profile with a tall and narrow peak, whereas the velocity profile of the 1000-ms movement is wider and flatter and nearly coincides with that predicted by the minimum-torque-change principle. correspondingly, the ratio of peak velocity to average velocity is 2.11 for the fast movement (100 ms) compared to 1.91 for the slow one (1000 ms).

From the above results, we see that the simultaneous minimization of costs associated with the first and the second time derivative of torque predicts trajectories whose peak-velocity to average-velocity ratio increases as movement duration decreases. This is consistent with experimental data showing that this ratio is about 1.8–1.9 for slow movements (Nagasaki, 1989) and about 2.0–2.2 for fast movements (Nagasaki, 1989; Wiegner & Wierzbicka, 1992). As discussed above, the minimum-torque-change principle incorrectly predicts that this ratio cannot exceed 1.875 (Engelbrecht & Fernández, 1997). With respect to this trajectory characteristic, our newly developed theory therefore makes more accurate predictions than the minimum-torque-change theory.

One may object, however, that the good performance of our new minimum principle is no surprise because it is apparent that the cost functional (25) and the value of λ were specifically chosen to produce such a god fit to the experimental data. This criticism is certainly valid, but, to a large extent, it also applies to other minimum theories. The minimum-torque-change principle, for instance, was proposed by Uno *et al.* (1989) based on their experimentation with a number of

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[10] If viscosity is negligible, as is assumed here, one-DOF horizontal-plane minimum-torque-change trajectories are independent of movement duration. For a treatment of the minimum-torque-change principle that includes viscosity, see Engelbrecht and Fernández (1997).

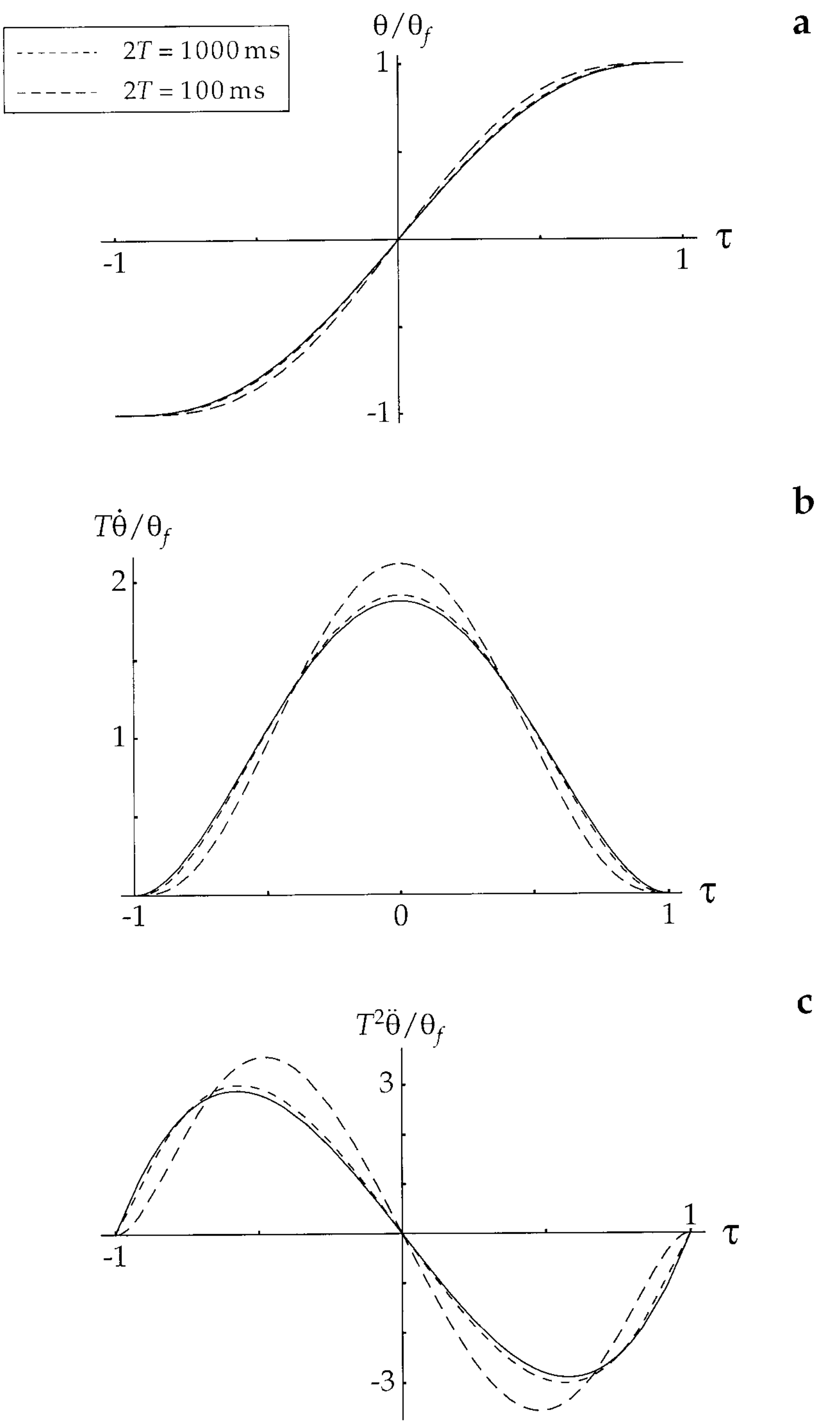

FIG. 4. Trajectories for one-DOF horizontal-plane movements that minimize the time derivative of torque (solid line) or a weighted sum ($\lambda = 0.1$ kHz) of the time derivative and the second time derivative of torque (dashed lines). (a) Position, (b) velocity, and (c) acceleration.

different minimum principles, and a similarly empirical approach was used by Yashin-Flash (1983) in her development of the minimum-jerk theory. The selection criterion for both minimum principles was the accuracy of their predictions relative to the considered alternatives.

Let us take the above approach one step further by considering the following procedure for theory selection: First, find the function that best describes arm movement trajectories. Second, find a cost functional with regard to which this trajectory function is optimal[11]. Obviously, this approach corresponds to the one caricatured by Gould and Lewontin (1979) and thus invites the same criticism of circular reasoning. But this criticism only applies to minimum principles that are used in the explanatory sense that is typical of evolutionary biology. If, on the other hand, one views a minimum principle as a purely descriptive tool that concisely summarizes a set of experimental data, as is the case in physics, nothing is wrong with this approach.

The performance measures that underlie most current minimum principles in motor control are specifically designed to predict experimental data as accurately as possible. A circular argument is therefore made if the observed behavior is, in turn, explained as an optimal adaptation with respect to these performance measures. To avoid such circular arguments, empirically-derived minimum principles are best viewed as abstract laws whose sole purpose it is to describe behavior in a compact mathematical form.

Critical Tests for Descriptive Minimum Theories

In the behavioral sciences, mathematical laws that are used in a purely descriptive fashion are not uncommon. Three famous examples are the Weber–Fechner law of subjective intensity perception (Fechner, 1860/1966), the logarithmic law of memory decay (Ebbinghaus, 1885/1964), and the logarithmic speed–accuracy tradeoff (Fitts, 1954). All three of these are functional relations. How do laws of this type relate to minimum principles?

If we consider only a single behavioral task with fixed boundary conditions, we find that there is generally a one-to-one relation between minimum principle and functional relation, so that the former appears to be no more than an unnecessarily complicated way of stating the latter. The situation is different, however, if a number of tasks are considered, each of which imposes a different behavioral constraint. In this case, a minimum principle may predict a different functional relation for each task. Minimum principles are therefore more general than functional relations and may be used to combine a number of these into a single unifying law.

Most experimental research in motor control has been concerned with unconstrained point-to-point reaching movements, which may be described quite

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[11] Formally, this amounts to solving an inverse problem of the calculus of variations. Rosen (1967, Chap. 5) gives some insight into how this can be done.

accurately by a variety of simple functional relations (Plamondon, Alimi, Yergeau, & Leclerc, 1993). To find a minimum principle that corresponds to one of these is by itself not a significant achievement. However, if such a minimum principle is then applied to a new task for which it predicts a different functional relation, and this new prediction is accurate as well, one may claim that an important discovery has been made.

For certain minimum principles, interesting or even surprising predictions may arise when tasks are considered that require a movement to be performed against an externally applied force or torque. We shall illustrate this for the minimum-torque-change principle. Consider a horizontal-plane one-DOF arm movement with negligible viscosity performed against an external torque that varies linearly with the arm's angular position. As in Eq. (26), the torque $N(t)$ that is generated by the muscles is proportional to the angular acceleration of the arm, but now an additional torque is necessary to oppose the external torque. Hence,

$$N(t) = I\ddot{\theta} + K\theta, \tag{32}$$

where K determines the magnitude of the external torque. We take the time derivative of Eq. (32) and square the resulting expression to obtain the instantaneous torque-change cost

$$\dot{N}^2 = I^2\dddot{\theta}^2 + K^2\dot{\theta}^2 + 2IK\dot{\theta}\dddot{\theta}. \tag{33}$$

Application of the Euler–Poisson equation to Eq. (33) then yields

$$\phi^{(4)} + 2\beta^2\ddot{\phi} + \beta^4\phi = 0, \qquad \phi \equiv \ddot{\theta}, \quad \beta \equiv \sqrt{K/I}, \tag{34}$$

a homogeneous fourth-order linear differential equation, whose solution takes the form

$$\phi(t) = a_2' \cos \beta t + a_3' \sin \beta t + a_4' t \sin \beta t + a_5' t \cos \beta t, \tag{35}$$

and after double integration, we have

$$\theta(t) = a_0 + a_1 t + a_2 \cos \beta t + a_3 \sin \beta t + a_4 t \sin \beta t + a_5 t \cos \beta t. \tag{36}$$

We find the coefficients a_0–a_5 from the boundary conditions

$$\theta(-T) = -\theta_f, \qquad \theta(T) = \theta_f, \qquad \dot{\theta}(\pm T) = \ddot{\theta}(\pm T) = 0 \tag{37}$$

and insert them into Eq. (36), thus obtaining the minimum-torque-change trajectory equation

$$\theta(\tau) = \theta_f \frac{(2\omega^2 + \omega \sin 2\omega)\, \tau - (2\omega \cos \omega + 4 \sin \omega) \sin \omega\tau + 2\omega(\sin \omega)\, \tau \cos \omega\tau}{2 \cos 2\omega + \omega \sin 2\omega + 2\omega^2 - 2}, \tag{38}$$

with $\omega \equiv T\beta = T\sqrt{K/I} > 0$ and $\tau \equiv t/T \in [-1, 1]$.

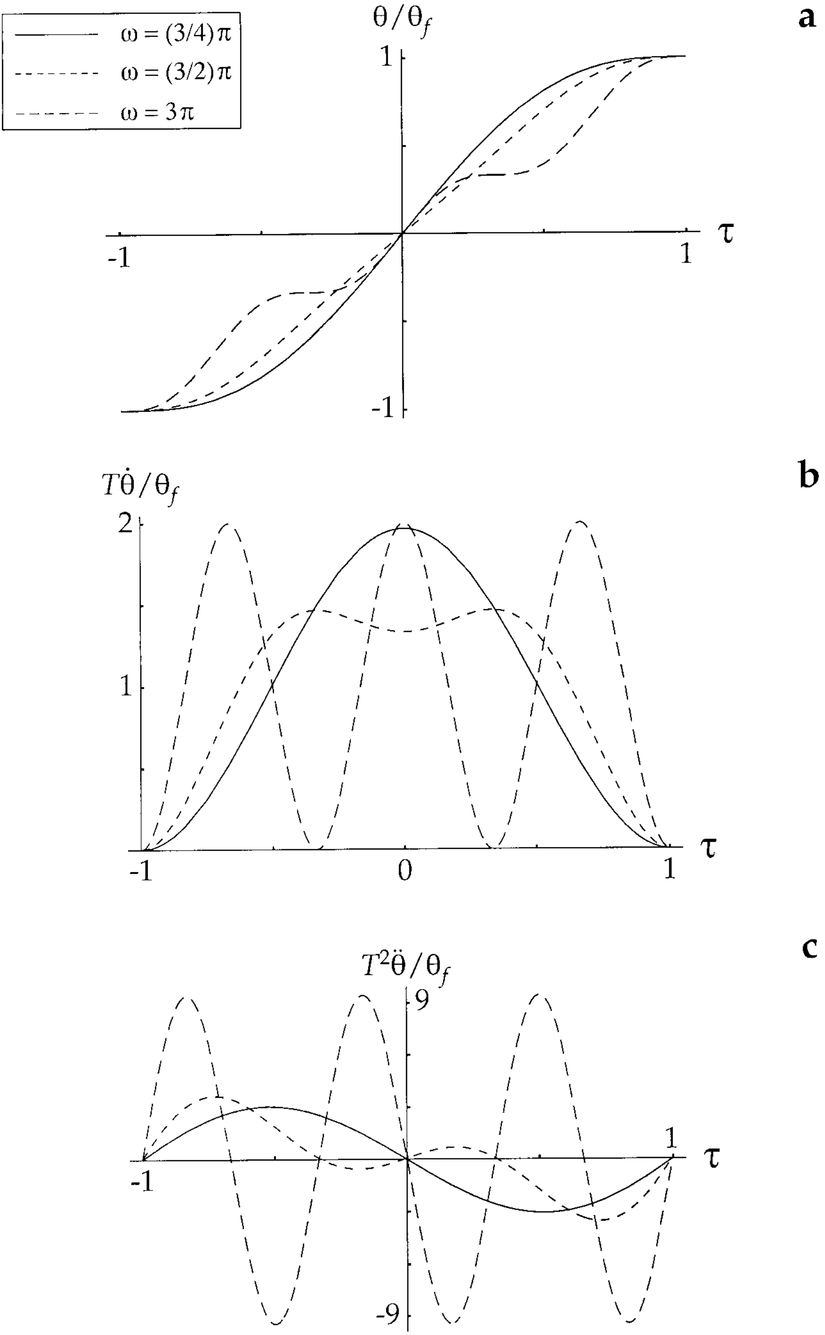

FIG. 5. One-DOF horizontal-plane minimum-torque-change trajectories for movements that are subjected to an externally-applied torque. The external torque decreases linearly with the angular position of the arm and changes sign halfway between the initial and target positions. (a) Position, (b) velocity, and (c) acceleration.

Equation (38) represents a family of trajectories that is parametrized by ω. Three members of that family, corresponding to $\omega = (3/4)\,\pi$, $\omega = (3/2)\,\pi$, and $\omega = 3\pi$, are shown in Fig. 5.

Note that the trajectory corresponding to $\omega = (3/4)\,\pi$ has a single-peaked velocity profile, as is typical for unconstrained point–point movements, while those corresponding to $\omega = (3/2)\,\pi$ and $\omega = 3\pi$ have double-peaked and triple-peaked velocity profiles, respectively. This result is quite interesting; it seems to suggest that the number of velocity extrema is proportional to ω and thus to the square root of K, the magnitude of the external torque. Careful analysis of Eq. (38) shows that this is indeed the case. In particular, we may assert the following:

THEOREM (External torque). *Let* $\Omega \equiv \omega/\pi$, *and let* $\lfloor \Omega \rfloor$ *denote the greatest integer not larger than* Ω. *Trajectories obeying Eq.* (38) *have exactly* $2\lfloor \Omega \rfloor + 3$ *velocity extrema and saddle points in the interval* $[-1, 1]$ *if* $\cot \omega < 1/\omega$ *and* $\omega \neq k\pi$, $k \in \mathscr{Z}$. *Otherwise, they have* $2\lfloor \Omega \rfloor + 1$ *velocity extrema and saddle points in the interval.*

A proof of this theorem is given in Appendix A.

We have chosen to present the above result because it exemplifies the form a critical test of the minimum-torque-change theory, or of any other minimum theory, should take. The predictions are so curious that they offer and ideal opportunity either to falsify the theory or, alternatively, to confirm its universal applicability.

It is also worth noting that the above result sets the minimum-torque-change principle in direct opposition to the minimum-jerk principle, which predicts complete independence of trajectory shape from movement dynamics. We therefore have identified a paradigm that gives us completely contradictory predictions for the two theories, and if these predictions are put to an experimental test, it is essentially guaranteed that at least one of the two theories will be refuted.

EXPLANATORY THEORIES RECONSIDERED

In motor control, minimum theories are often reverse engineered to fit certain sets of experimental data. Accordingly, their function is best viewed as a purely descriptive one. In evolutionary biology, on the other hand, a minimum theory is usually selected based on the relation of its underlying cost measure to the concept of biological fitness (but see Gould & Lewontin, 1979). The objective of such a minimum theory is not only to predict behavior but also to explain it as the result of optimal adaptation.

Two cost measures that are almost certainly related to biological fitness are the metabolic energy that is required to perform a movement and the time that it takes to perform it. Few biologists would doubt that animals are generally well adapted with respect to these measures[12]. Motor control theorists have also considered these measures (for mathematical treatments, see Nelson,1983; Oğuztöreli & Stein, 1983),

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[12] For a general discussion of low-pass filtering in the nervous system, see Partridge (1973).

but seem to have concluded that their minimization fails to predict the characteristics of human arm movement (for instance, Kawato, 1996; Uno *et al.* 1989). Indeed, the preference for the empirically-based development of minimum theories among motor control researchers is likely a result of these failures. However, the rejection of biologically-motivated minimum theories may have been premature. We shall demonstrate this for the minimum-time principle.

Minimum Time

Again, it will be instructive to consider the simplest possible case, that of one-DOF horizontal-plane movements. For these, the angular acceleration of the arm is a linear function of the torque that is applied to the joint. As discussed in Hermes and LaSalle (1969), minium-time control of such a linear system is always of the "bang–bang" type: The control variable (in the present case, torque) adopts only two values, its maximum and its minimum, and it switches from one to the other no more than $n-1$ times, where n is the order of the dynamical system.

If viscosity is negligible, the dynamics of a one-DOF planar-horizontal arm are described by the second-order Eq. (26) or, equivalently, by the system of two first-order equations

$$\frac{d}{dt}\theta=\dot{\theta}, \qquad \frac{d}{dt}\dot{\theta}=N/I. \tag{39}$$

We therefore have $n=2$, and the time-optimal control for the arm switches only once; it obeys

$$N(t)=\begin{cases} N_{\max} & t_0 \leqslant t < t_s \\ -N_{\max} & t_s \leqslant t \leqslant t_f, \end{cases} \tag{40}$$

where t_s denotes the switching time, and $N_{\max}$ is the largest admissible torque. If movement starts and ends at rest, it follows that the minimum-time trajectory is characterized by three acceleration discontinuities, one at the start, one at the switching point t_s, and one at the end. But even when subjects are instructed to move at maximum speed, their arm movement trajectories do not show such acceleration discontinuities (Wiegner & Wierzbicka, 1992). It thus seems that human arm movements are indeed not time-optimal.

This conclusion, however, depends on the constraints one is willing to impose on the set of functions from which $N(t)$ is to be chosen. Note that, in the above derivation, we assume that $|N(t)|$ is constrained to be no larger than $N_{\max}$. It is necessary to impose this constraint if one wants to avoid the absurd prediction that the time-optimal control must be composed of an infinitely short and infinitely tall positive torque pulse followed by an infinitely short and infinitely tall negative one, resulting in an infinitely fast arm movement. Most optimality theories in biology require such constraints. Otherwise, they would predict that organisms are infinitely strong, infinitely fecund, and infinitely long-lived (cf. Alexander, 1982, Chap. 6).

But once one accepts that $N(t)$ needs to be bounded, one should be willing to consider other constraints on it as well. It is to be expected that the general architecture of the neuromuscular control system limits the set of realizable torque functions in a number of ways. One way of incorporating these constraints is to develop a detailed neuromuscular model and to consider the input to this model (rather than torque) as the variable that is to be optimized. Hatze (1976) and Pandy, Zajac, Sim, and Levine (1990) models are instances of this approach. While this is a reasonable way to proceed, it is clear that this approach leads to a great increase in model complexity, and it thus sacrifices the simplicity and elegance of the minimum principles we have considered so far.

Some of this complexity may be unnecessary. This is so because many components of the neuromuscular controller do not impose any constraints on the set of realizable torque functions, which makes them irrelevant with regard to the optimal control solution. An alternative approach is, therefore, to restrict consideration to those features of the neuromuscular controller than obviously do impose constraints on the controller's output. One such constraint arises from the low-pass filtering property of muscle. When a muscle is electrically stimulated by a sinusoidal pulse of low frequency, the muscle generates contractions at a frequency that corresponds to that of the stimulus signal. But when the stimulus frequency is increased, these contractions become less and less forceful until they vanish altogether (Partridge, 1966). The muscle therefore responds selectively to low-frequency signals and ignores those of higher frequency; it acts as a low-pass filter.

In the nervous system, low-pass filtering not only occurs when a signal is transmitted from a motoneuron to a muscle; it also occurs whenever a signal is transmitted from one neuron to another[13] . The cause for this seems to be the temporal summation that occurs when a signal is transmitted across a synaptic junction. Mathematically, we may describe this by a convolution integral of the form

$$\varphi_1(t) = \int_0^t f(t)\, \varphi_0(\tau - t)\, d\tau, \tag{41}$$

where $\varphi_1(t)$ is the postsynaptic signal, $\varphi_0(t)$ is the presynaptic signal, and $f(t)$ is a weighting function. Often $f(t)$ decreases exponentially with time, and the convolution integral takes the form

$$\varphi_1(t) = -\ln \alpha \int_0^t \alpha^t \varphi_0(\tau - t)\, d\tau, \qquad \alpha \in (0, 1). \tag{42}$$

As we show in Appendix B, Eq. (42) instantiates a filter that is formally equivalent to a dynamical system whose behavior is governed by the following first-order linear differential equation:

$$\varphi_1(t) + \omega^{-1} \dot{\varphi}_1(t) = \varphi_0(t), \qquad \omega \equiv -\ln \alpha. \tag{43}$$

[13] For a general discussion of low-pass filtering in the nervous system, see Partridge (1973).

By extension, a signal that is submitted to a series of n such filters (as a result of being transmitted across a number of synapses) may be described by a system of n first-order linear differential equations:

$$\varphi_k(t) + \omega^{-1}\dot{\varphi}_k(t) = \varphi_{k-1}(t), \qquad k = 1, 2, ..., n. \tag{44}$$

What are the implications of these considerations for the minimum-time control of arm movement? To answer this question, let us assume that $\varphi_n(t)$ determines the torque input to the one-DOF arm whose dynamics are given by Eq. (39), and let us further assume that the only variable subject to explicit control is $\varphi_0(t)$. Since the dynamical system that is instantiated by Eqs. (39) and (44) is governed by a system of $n+2$ first-order linear differential equations, time-optimal control is still of the bang–bang type, but now the control has $n+1$ switches instead of just two, i.e.,

$$\varphi_0(t) = \varphi_{\max} \sum_{k=0}^{n+1} (-1)^k \left(\mathrm{H}(t - t_k) - \mathrm{H}(t - t_{k+1})\right), \tag{45}$$

where $\varphi_{\max}$ denotes the maximum admissible value for $|\varphi_0(t)|$, $\mathrm{H}(t)$ is the Heaviside step function, and the t_k are start, end, and switching times.

But even though the control signal $\varphi_0(t)$ is bang–bang and thus discontinuous, the torque command $\varphi_n(t)$ is a continuous function. As we show in Appendix C, the response of the nth-order filter (44) to a step input is

$$\hat{\varphi}_n(t) = \mathrm{H}(t)\left(1 - e^{-wt} \sum_{k=0}^{n-1} \frac{\omega^k}{k!} t^k\right), \tag{46}$$

assuming $\varphi_k(0) = 0$ for $k = 1, 2, ..., n$, and its response to $\varphi_0(t)$ follows from Eqs. (45) and (46) by application of the principle of superposition:

$$\varphi_n(t) = \varphi_{\max} \sum_{k=0}^{n+1} (-1)^k \left(\hat{\varphi}_n(t - t_k) - \hat{\varphi}_n(t - t_{k+1})\right). \tag{47}$$

As we can see from this expression, $\varphi_n(t)$ is continuous, and so is the acceleration of the arm because $\ddot{\theta}(t) = N(t)/I$ and $N(t) = \varphi_n(t)$. From the acceleration function, the minimum-time trajectory equation is then obtained by double integration and imposition of the initial conditions $\theta(0) = \theta_0$ and $\dot{\theta}(0) = 0$, which yield

$$\theta(t) = \theta_0 + \frac{\varphi_{\max}}{I} \sum_{k=0}^{n+1} (-1)^k \left(\hat{\theta}(t - t_k) - \hat{\theta}(t - t_{k+1})\right)$$

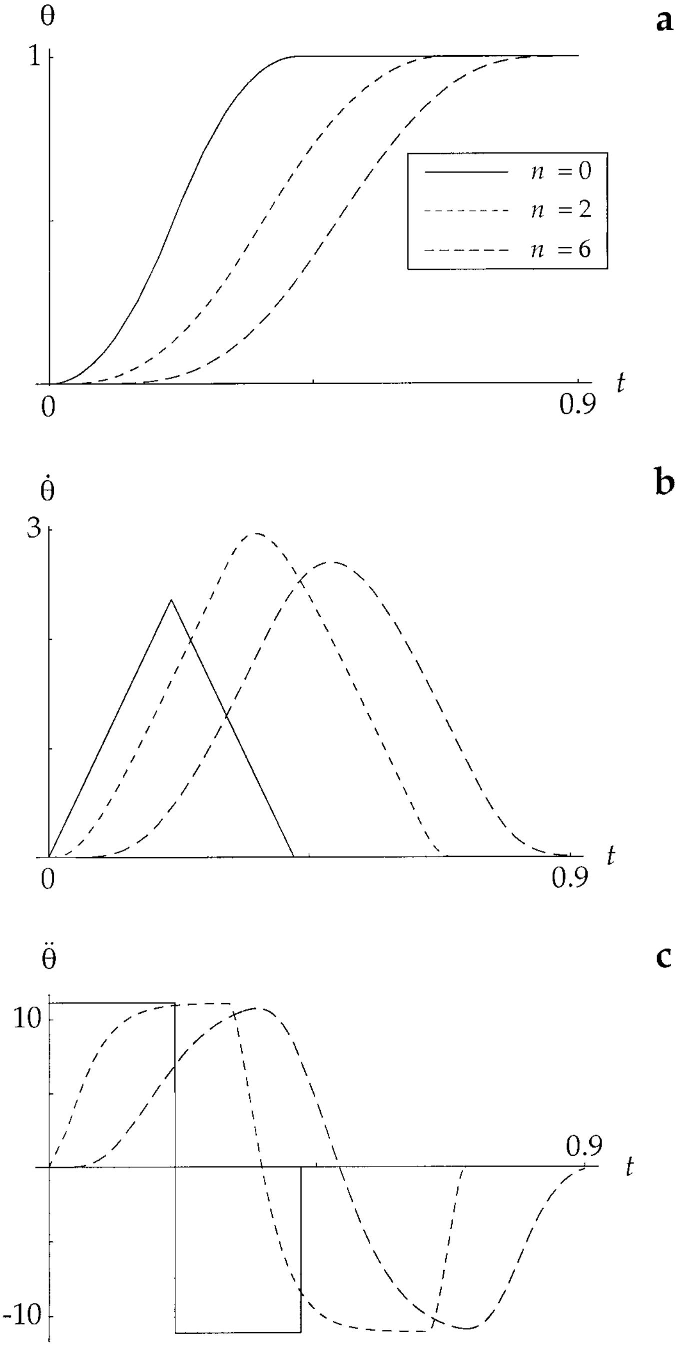

FIG. 6. One-DOF horizontal-plane minimum-time trajectories. The control signal generates torque after passing through zero (solid line), two (short dashes), or six (long dashes) first-order low-pass filters with $\omega = 30$. Time is given in seconds and angular position in radians.

with

$$\hat{\theta}(t) = \mathrm{H}(t)\left(\frac{t^2}{2} - \frac{n}{\omega}\,t + \frac{n(n+1)}{2\omega} - e^{-\omega t}\sum_{k=0}^{n-1}\frac{(n-k)(n+1-k)\,\omega^{k-2}}{2k!}\,t^k\right). \tag{48}$$

This equation has $n+2$ unknown coefficients, the switching times t_1 through t_{n+1} and the final time t_{n+2}. These must be found from the terminal boundary conditions

$$\theta(t_{n+2}) = \theta_f, \quad \dot{\theta}(t_{n+2}) = 0, \quad \varphi_k(t_{n+2}) = 0, \qquad k = 0, 1, \ldots, n. \tag{49}$$

Equation (48) represents a family of minimum-time trajectories that is parameterized by n, the number of filters that intervene between the control input $\varphi_0(t)$ and the torque command φ_n. Figure 6 shows three of these for systems with zero, two, and six filters. The trajectories corresponding to the two- and six-filter systems have continuous acceleration profiles, and the trajectory for the six-filter system is qualitatively quite similar to those observed in studies of rapid arm movement (Wiegner & Wierzbicka, 1992; Zelaznik, Schmidt, & Gielen, 1986). Our results therefore suggest that the minimum-time principle has indeed been rejected prematurely: Human arm movement may after all be time-optimal, at least within the constraints imposed by the low-pass filtering property of the nervous system.

STOCHASTIC MODELS

An important aspect of human movement that we have ignored so far is its stochastic nature. On the behavioral level, this is a well-established phenomenon. When participants are instructed to repeatedly produce identical movements of prescribed amplitude and duration, they instead produce a distribution of movements that is characterized by a scattering of endpoints whose standard deviation scales linearly with average movement speed (Schmidt *et al.*, 1979; Meyer, Smith, & Wright 1982); quite apparently, participants do not and cannot produce identical movements across a series of trials. In addition to the behavioral evidence, there is physiological data suggesting that stochasticity is, in fact, a fundamental characteristic of neural information processing (Calvin & Stevens, 1968; Clamann, 1969).

Despite these facts, it has been common practice to model human movement control as a deterministic rather than a stochastic process. Indeed, all minimum theories discussed so far have implicitly made this simplifying assumption. These theories have nonetheless been successful at describing important phenomena, such as the main kinematic characteristics of arm movement. Other phenomena, however, are difficult to explain within a deterministic framework. One example is the logarithmic speed–accuracy trade-off, known as Fitts's law (Fitts, 1954). Empirically, this law has proved extremely robust across a plethora of motor activities and experimental conditions (e.g., Kerr, 1973; Langolf, Chaffin, & Foulke, 1976; Meyer *et al.*, 1982). Its interpretation, however, has been highly controversial. Fitts's original information-theoretic explanation, which is based on a rather strained analogy between the motor system and a band-limited communication channel (Kvalseth, 1979), has generally not been taken very seriously (Meyer, Smith, Kornblum, Abrams, & Wright, 1990). An alternative theory, the deterministic iterative-corrections model put forward by Crossman and Goodeve (1963/1983), was more convincing, but several of its predictions turned out to be inconsistent with experimental data (Meyer *et al.*, 1990).

Currently, the best explanation of Fitts's law is offered by the stochastic optimized-submovement model (Meyer, Abrams, Kornblum, Wright, & Smith, 1988). This model assumes that a reaching movement may consist of two movement units, an initial and a corrective one. The endpoint of the initial movement is a random variable that follows a normal distribution with mean at the center of the target and standard deviation proportional to average movement speed. If, due to its stochastic nature, the initial movement misses the target, a corrective movement is initiated. The endpoint of this movement is also a normal random variable, again with mean at the center of the target and standard deviation proportional to the unit's average speed. The speeds of the two movement units are selected such that expected total movement duration is minimized, subject to the constraint that the corrective movement, when invoked, has a fixed probability of terminating on the target. With speeds chosen in this fashion, one obtains a speed–accuracy trade-off that follows a square-root law. This law is similar to Fitts's law (which predicts a logarithmic trade-off), and it may actually be somewhat more accurate than the latter (Kvalseth, 1980).

At least one of the assumptions of the Meyer *et al.* (1988) model is, however, questionable: Empirically, it has been found that the mean amplitude of the initial movement unit is not always equivalent to target distance. Indeed, for saccadic eye movements, it is well established that the initial saccade systematically undershoots the target by an average of 5 to 10% of movement amplitude (Becker, 1989; Henson, 1978), and systematic undershoots are also known to occur in high-precision arm movement (Milner, 1992). In infants, this *undershoot bias* is even more pronounced; for infants less than 20 weeks of age, systematic undershoots of 50% of total amplitude have been reported, both for arm movement (Hofsten, 1991) and for saccadic eye movement (Aslin & Salapatek, 1975).

It is therefore a natural extension of the stochastic optimized-submovement model to make the amplitudes of the movement units part of the optimization process as well. Using this approach, Harris (1995) showed that, given the typical endpoint variability of saccades, it is time-optimal for initial saccades to undershoot the target by about 5 to 10% of total amplitude. Berthier (1996) extended this analysis to infant reaching movements, which are characterized by multiple movement units and large initial-unit undershoots (Hofsten, 1991). With increasing age, the number of these units decreases, the amplitude of the initial unit increases, and the speed of the individual units follows an N-shaped curve (increase–decrease–increase). Berthier showed that this complex developmental pattern can be reproduced if one assumes that infants behave according to a stochastic time-optimal strategy that is continuously adapted to the level of motor noise, which decreases with the infant's age. His results suggest that the motor control system is capable of optimally adapting its control policies to changing levels of stochasticity.

Further Refinements

So far, a rather important consequence of the stochastic nature of human motor control has been ignored: In combination with information delays due to slow transmission and processing of perceptual information, stochastic control leads to incomplete observability of state information (Engelbrecht & Katsikopoulos, 1999). (Note that proprioceptive delays are at least 30 ms, and visual delays easily exceed 100 ms.) This is so because delayed perceptual information is not sufficient to define the motor system's current state with certainty, even if it is augmented by information about the intervening motor commands. The most one can obtain from the available information is a probability distribution over current states, and this significantly alters the nature of the control problem (cf. Sondik, 1978). The implications of this for the optimal control of motor behavior are far-reaching, and their investigation is an important research topic for the future.

SUMMARY AND CONCLUSIONS

Minimum principles may contribute to our understanding of motor performance in two ways, as descriptive tools and as explanatory ones. As descriptive tools, they provide concise mathematical summaries of large sets of experimental data, and as explanatory tools, they interpret motor performance as the result of an optimal adaptation with respect to some (biologically relevant) efficiency criterion.

However, if the efficiency criterion itself is selected to obtain a maximally accurate fit to the behavioral data, as has been the case for most minimum principles in motor control, the explanatory function becomes tautological. Minimum principles of this kind should therefore be used as purely descriptive tools and not as explanatory ones.

Some minimum principles, such as the minimum-jerk principle (Flash and Hogan, 1985) and the minimum-torque-change principle (Uno *et al*., 1989), have been quite successful at summarizing the gross characteristics of point-to-point arm movements and have accordingly enjoyed considerable popularity. However, point-to-point arm movements are also well described by a number of simple functional relations. In fact, if a minimum principle is applied to a single type of task, there is usually a one-to-one correspondence between functional relation and minimum principle, so that the former is no more than an arcane restatement of the latter. Critical tests of minimum principles should therefore involve tasks that are sufficiently different from those that prompted their formulation. For each such task, a minimum principle may predict a different functional relation, and some of these predictions may turn out to be surprising or even counterintuitive.

Most current minimum principles in motor control have been chosen empirically, based on their success at predicting the characteristics of arm movement, and biological relevance has often only been considered as an afterthought. Efficiency criteria such as the minimization of movement time and the minimization of metabolic energy requirements, which biologists believe to be of great adaptive value, have generally been ignored, apparently because they seem to predict movement trajectories that are inconsistent with experimentally observed ones. However, this conclusion is only valid if we assume that the neural control signal may be any function of bounded magnitude. Imposition of additional constraints upon the control signal may lead to entirely different predictions, as we demonstrated for the minimum-time principle: If the low-pass filtering property of neural signal processing is admitted as a constraint upon the set of physically realizable torque functions, theoretical minimum-time trajectories are qualitatively quite similar to those observed in rapid arm movement.

Another property of the neuromuscular control system that must be dealt with is the stochastic nature of control. This property is important because control strategies that are optimal for a deterministic system are often nonoptimal for the corresponding stochastic system and vice versa. For instance, while Fitts's law and the undershoot bias are difficult or impossible to explain within a deterministic optimal control framework, straightforward interpretations are possible once stochasticity is explicitly taken into consideration.

For future research, it will be an important task to determine which properties of the neuromuscular control system make a difference with regard to the optimal control of movement. Considerable progress in our understanding of motor behavior can be expected from the identification of these properties and from their inclusion in minimum theories.

APPENDIX A

Proof of the External-Torque Theorem

A necessary and sufficient condition for a velocity extremum or saddle point is $d^2\theta/d\tau^2 = 0$. From Eq. (38), we obtain

$$\frac{d^2}{d\tau^2}\theta(\tau) = 2\omega^3\theta_f\left(\frac{\cos\omega\sin\omega\tau - (\sin\omega)\,\tau\cos\omega\tau}{2\cos 2\omega + \omega\sin 2\omega + 2\omega^2 - 2}\right), \qquad \tau \in [-1, 1]. \tag{A.1}$$

Hence, any τ that is the location of a velocity extremum or saddle point must be a solution to the equation

$$\cos\omega\sin\omega\tau = (\sin\omega)\,\tau\cos\omega\tau. \tag{A.2}$$

If $\omega = j\pi$, $j = 1, 2, \ldots$, the above equation reduces to

$$\sin\omega\tau = 0, \tag{A.3}$$

which is solved by any $\tau = k\Omega^{-1}$, $k \in \mathscr{Z}$. Since $\tau \in [-1, 1]$, we have $k \in [-\Omega, -\Omega + 1, ..., \Omega - 1, \Omega]$, so that there are a total of $2\Omega + 1$ solutions.

If $\omega \neq k\pi$, we may divide Eq. (A.2) by $\sin \omega$ to obtain

$$\cot \omega \sin \omega\tau = \tau \cos \omega\tau. \tag{A.4}$$

Three obvious solutions of this equation are $\tau = 0$ and $\tau = \pm 1$. It appears impossible to find the other solutions of this equation by analytical means, but we shall nevertheless be able to determine their number. To simplify Eq. (A.4), we divide both sides of it by $\sin \omega\tau$ (note that $\sin \omega\tau = 0$, $\tau \neq 0$, is not a solution) to obtain

$$\cot \omega = \tau \cot \omega\tau. \tag{A.5}$$

This equation has exactly $2\lfloor \Omega \rfloor + 3$ solutions if $\cot \omega < 1/\omega$; otherwise, it has $2\lfloor \Omega \rfloor + 1$ solutions. To prove this, we need to consider the following properties of the cotangent function,

$$\lim_{\tau \to 0} \tau \cot \omega\tau = 1/\omega, \tag{A.6}$$

$$\lim_{\tau \to (k/\Omega)-} \tau \cot \omega\tau \to -\infty, \qquad k = 1, 2, ..., \tag{A.7}$$

$$\lim_{\tau \to (k/\Omega)+} \tau \cot \omega\tau \to \infty, \qquad k = 1, 2, ..., \tag{A.8}$$

$$\frac{d}{d\tau} \tau \cot \omega\tau = \csc^2 \omega\tau(\cos \omega\tau \sin \omega\tau - \omega\tau) < 0 \qquad \text{if} \quad \tau > 0. \tag{A.9}$$

We invoke the intermediate-value theorem to conclude from the above properties that $\tau \cot \omega\tau$ takes on all values between $-\infty$ and $1/\omega$ for $\tau \in (0, 1/\Omega)$ and that it takes on all values between $-\infty$ and ∞ for $((k-1)/\Omega, k/\Omega)$, $k = 2, 3, ...$. From Eq. (A.9), we know that in each of these intervals the mapping $\tau \to \tau \cot \omega\tau$ is one-to-one. Hence, Eq. (A.4) has exactly one solution in the interval $(0, 1/\Omega]$ if $\cot \omega < 1/\omega$; otherwise it has no solution in that interval. It also follows from above that Eq. (A.4) has exactly one solution in each interval $[(k-1), \Omega, k/\Omega]$, $k = 2, 3, ..., \lfloor \Omega \rfloor$. Finally, we note that $(\lfloor \Omega \rfloor/\Omega, 1] \subseteq (\lfloor \Omega \rfloor/\Omega, \lfloor \Omega + 1 \rfloor/\Omega]$ implies that the obvious solution $\tau = 1$ is the only one in the interval $(\lfloor \Omega \rfloor/\Omega, 1]$. Combining all of the above results, we conclude that Eq. (A.4) has $\lfloor \Omega \rfloor + 1$ solutions in the interval $(0, 1]$ if $\cot \omega < 1/\omega$; otherwise, it has $\lfloor \Omega \rfloor$ solutions in that interval. Since $d\theta(\tau)/d\tau$ is even, the number of extrema and saddle points in $[-1, 0)$ corresponds to the one in $(0, 1]$. In conjunction with the obvious solution $\tau = 0$, this yields a total number of $2\lfloor \Omega \rfloor + 3$ extrema and saddle points if $\cot \omega < 1/\omega$ and $2\lfloor \Omega \rfloor + 1$ extrema and saddle points if $\cot \omega \geqslant 1/\omega$.

APPENDIX B

Dynamical Systems Representation of a Linear Filter

Let us consider the following two expressions:

$$\varphi_1(t) + \omega^{-1}\dot{\varphi}_1(t) = \varphi_0(t), \qquad \omega \equiv -\ln \alpha, \tag{B.1}$$

$$\varphi_1(t) = -\ln \alpha \int_0^t \alpha^t \varphi_0(\tau - t)\, d\tau. \tag{B.2}$$

Application of the convolution theorem to Eq. (B.2) yields

$$\begin{aligned}\mathscr{L}\{\varphi_1(t)\} &= -\ln \alpha \mathscr{L}\{\alpha^t\}\ \mathscr{L}\{\varphi_0(t)\}\\ &= \frac{-\ln \alpha}{s - \ln \alpha} \mathscr{L}\{\varphi_0(t)\},\end{aligned} \tag{B.3}$$

where $\mathscr{L}\{\varphi(t)\}$ denotes the Laplace transform of $\varphi(t)$. Substituting $\omega \equiv -\ln \alpha$ and rearranging terms, we may rewrite this as

$$(1 + s\omega^{-1})\ \mathscr{L}\{\varphi_1(t)\} = \mathscr{L}\{\varphi_0(t)\}. \tag{B.4}$$

Assuming that the system is at rest at $t_0 = 0$, we have $s\mathscr{L}\{\varphi_1(t)\} = \mathscr{L}\{\dot{\varphi}_1(t)\}$, and inserting this into Eq. (B.4), we obtain

$$\mathscr{L}\{\varphi_1(t)\} + \omega^{-1}\mathscr{L}\{\dot{\varphi}_1(t)\} = \mathscr{L}\{\varphi_0(t)\}, \tag{B.5}$$

which is also the Laplace transform representation of Eq. (B.1). Hence, Eqs. (B.1) and (B.2) are equivalent.

APPENDIX C

Response of an *N*th Order Linear Filter to a Step Input

Let the Heaviside step function $\mathrm{H}(t)$ be the input to the linear filter defined by (44). The response of the filter is then determined by the following set of n first-order linear differential equations:

$$\varphi_1(t) + \omega^{-1}\dot{\varphi}_1(t) = \mathrm{H}(t), \tag{C.1}$$

$$\varphi_k(t) + \omega^{-1}\dot{\varphi}_k(t) = \varphi_{k-1}(t), \qquad k = 2, 3, ..., n. \tag{C.2}$$

We solve these by application of the Laplace transform. We let $t_0=0$, for mathematical convenience, so that $\varphi_k(0)=0$, assuming the system is initially at rest. We then have

$$\mathscr{L}\{\varphi_1\}=\frac{1}{s(\omega^{-1}s+1)}, \tag{C.3}$$

$$\mathscr{L}\{\varphi_k\}=\frac{\mathscr{L}\{\varphi_{k-1}\}}{w^{-1}s+1}, \qquad k=2,3,\ldots,n, \tag{C.4}$$

and by recursive application of Eq. (C.4), it follows that

$$\mathscr{L}\{\varphi_n\}=\frac{1}{s(w^{-1}s+1)^n}. \tag{C.5}$$

To find the inverse Laplace transform of this expression, we rewrite it as

$$\mathscr{L}\{\varphi_n\}=\frac{\omega^n}{s(s+\omega)^n}=\frac{\omega^n}{(n-1)!}\,\frac{(n-1)!}{(s+\omega)^n}\,\frac{1}{s} \tag{C.6}$$

and let

$$F(s)\equiv\frac{\omega^n}{(n-1)!}\,\frac{(n-1)!}{(s+\omega)^n}, \qquad G(s)\equiv\frac{1}{s}, \tag{C.7}$$

so that

$$\mathscr{L}\{\varphi_n\}=F(s)\,G(s). \tag{C.8}$$

The inverse Laplace transforms of $F(s)$ and $G(s)$ are

$$f(t)=\mathscr{L}^{-1}\{F(s)\}=\frac{\omega^n}{(n-1)!}\,t^{n-1}e^{-\omega t}, \tag{C.9}$$

$$g(t)=\mathscr{L}^{-1}\{G(s)\}=\mathrm{H}(t), \tag{C.10}$$

and the inverse Laplace transform of their product may be obtained from the convolution theorem:

$$\mathscr{L}^{-1}\{F(s)\,G(s)\}=\int_0^t f(\tau)\,g(t-\tau)\,d\tau. \tag{C.11}$$

Thus

$$\varphi_n(t)=\int_0^t f(\tau)\,\mathrm{H}(t-\tau)\,d\tau, \tag{C.12}$$

and since this expression vanishes for $t<0$, we may rewrite it as

$$\varphi_n(t) = \mathrm{H}(t) \int_0^t f(\tau)\, d\tau. \tag{C.13}$$

Replacement of $f(\tau)$ with its definition (C.9) and integration by parts then yield

$$\begin{aligned}\varphi_n(t) &= \mathrm{H}(t) \frac{\omega^n}{(n-1)!} \int_0^t \tau^{n-1} e^{-\omega\tau}\, d\tau \\ &= \mathrm{H}(t) \left(1 - e^{-wt} \sum_{k=0}^{n-1} \frac{\omega^k}{k!} t^k\right).\end{aligned} \tag{C.14}$$

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